Great Thinkers, Great Theorems

William Dunham, Ph.D.

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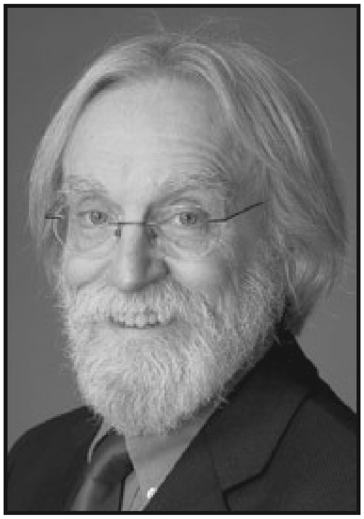

William Dunham, Ph.D.

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Professor William Dunham is the Truman Koehler Professor of Mathematics at Muhlenberg College. His undergraduate degree is from the University of Pittsburgh (1969), where he earned membership in Phi Beta Kappa during his junior year and received the M. M. Culver Award as Pitt's outstanding mathematics major in his senior year. From there, he went to The Ohio State University as a University Fellow from 1969 to 1974. At Ohio State, he finished his M.S. in 1970 and his Ph.D. in 1974, with a dissertation in general topology written under Professor Norman Levine.

Professor Dunham has taught at Hanover College in Indiana as well as at Muhlenberg College. He has received teaching awards from both institutions as well as the Award for Distinguished College or University Teaching from the Eastern Pennsylvania and Delaware Section of the Mathematical Association of America. In addition, he has twice been a visiting professor: first at Ohio State from 1987 to 1989; then, in the fall of 2008, at Harvard University, where he was invited to teach an undergraduate course on the work of the great Swiss mathematician Leonhard Euler.

In 1983, Professor Dunham received a summer grant from the Lilly Endowment to develop a "great theorems" course on the history of mathematics. This led not only to the class itself but to his first book, *Journey through Genius: The Great Theorems of Mathematics* (John Wiley and Sons, 1990), which became a Book-of-the-Month Club selection and has since been translated into Spanish, Italian, Japanese, Korean, and Chinese. Another spin-off was a series of summer seminars funded by the National Endowment for the Humanities (NEH) and directed by Professor Dunham at Ohio State in 1988, 1990, 1992, 1994, and 1996. As mathematics seminars, these were something of a departure from the usual NEH fare, but Professor Dunham's idea of portraying great theorems as works of (mathematical) art carried the day.

In the wake of that first book came two more in the 1990s—*The Mathematical Universe: An Alphabetical Journey through the Great Proofs, Problems, and Personalities* (John Wiley and Sons, 1994) and *Euler: The Master of Us All* (Mathematical Association of America, 1999). In the present millennium, Professor Dunham wrote a fourth book, *The Calculus Gallery: Masterpieces from Newton to Lebesgue* (Princeton University Press, 2005), and edited a fifth, *The Genius of Euler: Reflections on His Life and Work* (Mathematical Association of America, 2007). These books garnered various honors: *The Mathematical Universe* was designated by the Association of American Publishers as the outstanding mathematics book of 1994; *Euler: The Master of Us All* received the Mathematical Association of America's Beckenbach Book Prize in 2008; and both that volume and *The Calculus Gallery* were listed by *Choice* magazine among the outstanding academic titles of their respective years.

In addition to these books, Professor Dunham has written a number of articles on mathematics and its history. Among these are papers that received the George Pólya Award in 1993, Trevor Evans Awards in 1997 and 2008, and the Lester R. Ford Award in 2006. These awards, presented by the Mathematical Association of America, recognize excellence in mathematical exposition. In addition, Professor Dunham has provided mathematical journals with a cartoon ("Math Prodigy Field Trip") and a poem ("For Whom Nobel Tolls"), although these are unlikely to challenge the reputations of Charles Schultz or Emily Dickinson.

Over the years, Professor Dunham has presented numerous talks on mathematics and its history. These include lectures to students and faculty at scores of U.S. colleges and universities, ranging from Amherst to Bowdoin to Carleton, from Davidson to Denison to Dickinson. He has also addressed the scientific staff at businesses (Texas Instruments, Air Products) and governmental agencies (Goddard Space Flight Center, the National Institute of Standards and Technology), and he has performed on a larger stage with appearances on the BBC, on National Public Radio's *Talk of the Nation* "Science Friday," and at the Smithsonian Institution. Perhaps his most unusual venue was the Swiss Embassy in Washington DC, where Professor Dunham gave a 2007 lecture on Euler, a son of Switzerland whose tercentenary was being celebrated.

Professor Dunham is pleased to add to his resume this course for The Great Courses. As he has done throughout his career, he is happy to share the genius of great mathematicians, and the beauty of their great theorems, with a new audience. ■

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SUPPLEMENTAL MATERIAL

Great Thinkers, Great Theorems

Scope:

In this course, we meet some of history's foremost mathematicians and examine the discoveries that made them famous. The result is something of an adventure story. As with all such stories, the characters are interesting, if a bit eccentric. But rather than leading us to unexplored corners of the physical world, these adventurers will take us on a journey into the mathematical imagination.

Everyone is aware of the utility of mathematics. Its practical applications run from engineering to business, from astronomy to medicine. Indeed, modern life would be impossible without applied mathematics. No human pursuit is more useful. But the subject has another, more aesthetic side. It was the 20th-century philosopher Bertrand Russell who described mathematics as possessing "… not only truth but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show." It is *this* vision of mathematics as an unmatched creative enterprise that guides us as we explore the genius of great thinkers and the beauty of their great theorems.

We begin with mathematicians of ancient times. Chief among these are Euclid and Archimedes, whose specialty was geometry and whose achievements, even after two millennia, remain as fresh as ever. It was Euclid, of course, who gave us the *Elements*, the most successful, influential mathematics text of all time. His successor, Archimedes, stands as the most creative mathematician of the classical era for determining, among other things, the area of a circle and the volume of a sphere. Additionally, we meet Thales, Pythagoras, and Heron—three other Greek geometers who left deep footprints. Together, these individuals should give a sense of the rich mathematical tradition of that distant era.

With the fall of Rome, the world's intellectual center shifted eastward; thus, we next consider the justly esteemed Muhammad Mūsa ibn al-Khwārizmī, who in the 9th century, solved quadratic equations by (literally) completing the square. Then we follow the trail back to Renaissance Italy with the colorful Gerolamo Cardano, who published the algebraic solution of the cubic equation in 1545. This brings us to Europe in the 1600s, sometimes called the "heroic century" of mathematics. It was a time that saw the appearance of logarithms, number theory, probability, analytic geometry, and by century's end, the calculus. A list of individuals responsible for this explosion of knowledge reads like a scholar's hall of fame—Fermat, Descartes, Pascal, Newton, and Leibniz—each of whom we shall get to know.

Building on these achievements, the irascible Bernoulli brothers and the incomparable Leonhard Euler pushed the mathematical envelope throughout the 18th century. And the discoveries kept coming in the 19th, with the work of Gauss, Cauchy, Weierstrass, and Germain. We end with two lectures on Georg Cantor, who did battle with the mathematical infinite in what proved to be a shocking departure from all that had come before.

Besides meeting these remarkable characters, we should come to appreciate the masterpieces they left behind. These are the "great theorems," which are to mathematics what the "great paintings" are to art or the "great novels" are to literature. We shall consider some of these in full mathematical detail, among them, Euclid's proof of the infinitude of primes, Archimedes's determination of the area of a circle, Cardano's solution of the cubic, Newton's generalized binomial expansion, Euler's resolution of the Basel problem, and Cantor's theory of the infinite. An understanding of such works will reveal the extraordinary level of creativity that is required to produce a mathematical masterpiece.

This mathematical/historical/biographical journey will carry us through the centuries and across frontiers of the imagination. In the end, our course should provide an appreciation of the artistry of mathematics, a subject that Bertrand Russell aptly characterized as the place "…where true thought can dwell as in its natural home." ■

Theorems as Masterpieces

Lecture 1

It is a wonderful thing that progress in mathematics does not come at the expense of the past. In that sense, math differs from so many other subjects, so many other fields.

Rather than a course that teaches mathematical skills, this lecture series is a journey through the history of mathematics, focusing on the foremost mathematicians of all time—the great thinkers—and the extraordinary masterpieces they produced—the great theorems. We will regard theorems as the products of the creative imagination of mathematicians, and we will judge them by certain characteristics: elegance, or economy, and an element of unexpectedness, or surprise.

In mathematics, we will find that great theorems are not generally superseded by new discoveries or advances in the field. If a theorem is proved once, it is proved forever, and we will see results from ancient Greece that were still being used in 18th- and 19th-century mathematics and are cited today. In math, we do not discard older ideas; we build ever upward.

If a theorem is proved once, it is proved forever.

There are two general lines of attack for proving a theorem: direct and indirect proof. With direct proof, we reason directly from a hypothesis to a conclusion. Indirect proof might also be called proof by contradiction. With this strategy, we assume that the hypothesis is true, but the conclusion is false. From that assumption, we reason our way to a logical contradiction. If we reach such a contradiction, we may presume that the conclusion we were trying to prove is true. When a proof is complete, we end it with "Q.E.D.," the abbreviation for the Latin *quod erat demonstrandum* ("which was demonstrated").

One of the logical issues we will explore in this course is the contrast between a statement and its converse. Consider the statement: If *A*, then *B*. The converse reverses the role of the hypothesis and the conclusion: If *B*, then *A*. In mathematics, if we prove a statement, we almost always flip it around and ask: What about the converse? It is not always the case, however, that if a theorem is true, its converse is also true. As we will see in a future lecture, the converse of the Pythagorean theorem is not true in all cases.

We will begin the course with a lecture on pre-Euclidean mathematics before turning to three lectures on Euclid himself and his *Elements*, one of the greatest mathematics textbooks ever written. From there, we will move to Archimedes and his formula for finding the area of a circle. Then, we will jump to medieval Islam, to the world of Muhammad ibn Mūsā al-Khwārizmī, an Arabic scholar who wrote a well-known treatise on algebra. Next, we will meet Gerolamo Cardano, a strange character from 16th-century Italy who published the first proof of the solution of the cubic equation. In the 17th century, we will meet Isaac Newton and Gottfried Leibniz, co-creators of the calculus. Toward the end of the course, we will encounter Leonhard Euler, the most prolific mathematician in history, and his successor Carl Friedrich Gauss. We will conclude with Georg Cantor in the 19th century, who gave us the theory of the infinite, a profound, radical, and exciting idea.

As we begin our journey in the world of mathematics, Bertrand Russell's characterization seems particularly apt: "Remote from human passions, remote even from the pitiful facts of nature, the generations have gradually created an ordered cosmos where pure thought can dwell as in its natural home." ■

Suggested Reading

Hardy, *A Mathematician's Apology*.

Questions to Consider

1. A valid theorem about whole numbers is: "If m and n are even, then $m + n$ is even." State the converse and determine whether or not it is valid.

2. Prove the following theorem: "If the perimeter of a triangle is 35 feet, then at least one of its sides must be longer than 11.6 feet." HINT: Do this by contradiction; that is, begin by assuming that the conclusion is false.

Mathematics before Euclid

Lecture 2

Let me quote Richard Trudeau, a math author who said this: "… when the pall of familiarity lifts, as it occasionally does, and I see the Theorem of Pythagoras afresh, I am flabbergasted." It would be nice to try to retain that sense of wonder as we approach this great result. It is quite amazing.

Before we launch into Euclid's *Elements*, we take a brief look at the robust mathematical traditions of three non-Greek civilizations, Egypt, Mesopotamia, and China. We also meet two Greek mathematicians who predated Euclid, Thales and Pythagoras.

The Moscow Papyrus, preserved from ancient Egypt (c. 1850 B.C.), contains a theorem for finding the volume of a particular frustum of a pyramid. (A frustum is the object formed if we imagine slicing off the peak of a pyramid with a plane that is parallel to the pyramid's base.) Three important observations have been made about this papyrus: (1) The original scribe had to guess at the correct answer for this particular frustum. (2) The idea of attacking a general problem rather than a specific one seems to have been foreign to the Egyptians. (3) The papyrus includes no suggestion about why the given formula works; it is presented merely as a recipe.

The civilization of Mesopotamia, like Egypt, also had a rich mathematical tradition, often focused on the properties of whole numbers. Much of the mathematics of Mesopotamia is known to us from clay tablets that have survived through the centuries. A document called the *Chou Pei Suan Ching* gives us an example of ancient Chinese mathematics.

The idea of proving a general mathematical result originated with the Greeks, and according to tradition, the first mathematician to prove theorems was Thales. Among the theorems attributed to Thales are the following: (1) The base angles of an isosceles triangle are equal (congruent), (2) the sum of the angles of any triangle is two right angles (180 degrees), and (3) an angle inscribed in a semicircle is a right angle.

Of course, Pythagoras is another well-known mathematician from before the time of Euclid. He was born on the island of Samos but later moved to Crotona in southern Italy to establish a kind of think tank or early university. There, Pythagoras and his followers studied music, astronomy, and mathematics, believing that an understanding of mathematics would lead to greater understanding of the world.

The idea of proving a general mathematical result originated with the Greeks.

The Pythagorean theorem states that the square on the hypotenuse of a right triangle is the sum of the squares on the other two sides. We write this as $c^2 = a^2 + b^2$, but the Greeks did not think of the theorem as an equation with exponents; they thought of it as literal squares—squares built on the hypotenuse and the legs of the triangle. Looking at this idea pictorially, we can see that the area of the square built on the hypotenuse is the sum of the areas of the squares built on the legs. In other words, the Greeks thought about this theorem in terms of areas of squares.

It's likely that Pythagoras proved his theorem by starting with two squares of the same size, then dividing the squares up strategically and adding the areas of the resulting shapes. Dividing the squares differently, we get $a^2 + b^2$ + four triangles = c^2 + four triangles; we then subtract the triangles to get $a^2 + b^2 = c^2$. If we don't like algebra, we can also prove the Pythagorean theorem geometrically. In fact, there are hundreds of proofs of this theorem, many collected in a book called *The Pythagorean Proposition* by Elisha Scott Loomis. Just as artists are compelled to paint different landscapes, so are mathematicians compelled to make their mark by proving this result in unique ways. ■

Suggested Reading

Heath, *A History of Greek Mathematics*.

Joseph, *The Crest of the Peacock*.

Katz, ed., *The Mathematics of Egypt, Mesopotamia, China, India, and Islam*.

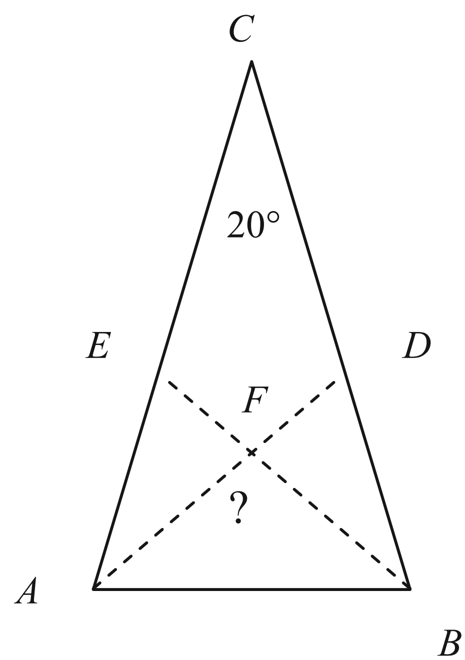
Loomis, *The Pythagorean Proposition*.

Robson, *Mathematics in Ancient Iraq*.

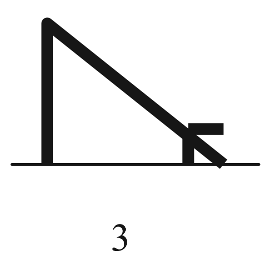
Swetz and Kao, *Was Pythagoras Chinese?*

Questions to Consider

1. For ΔABC, suppose AC and BC are of the same length and that the (degree) measure of $\angle ACB = 20°$ Suppose further that AD bisects $\angle CAB$ and BE bisects $\angle CBA$, with the two bisectors meeting at F. Using results attributed to Thales, find the degree measure of $\angle AFB$.



2. Here's a problem from an ancient Chinese text: A bamboo shoot 10 ch'ih tall is broken. The main shoot and its broken portion form a triangle. The top touches the ground 3 ch'ih from the stem. What is the length of the stem that is left standing erect? HINT: As noted in the lecture, the Chinese knew the Pythagorean theorem.



The Greatest Mathematics Book of All

Lecture 3

After you've done the first four propositions, proposition I.5 is a little bit more challenging, and some people, namely, the mathematical asses, could not cross this bridge, could not get over the *pons asinorum* and, thus, enter the remainder of Euclid's *Elements*.

We know little about the life of Euclid, except for the fact that he was the leading mathematician at the great Library of Alexandria in Egypt. Somewhere around 300 B.C., he wrote the *Elements*, a vast compendium of mathematics broken into 13 books and containing 465 propositions, or theorems. The *Elements* was highly regarded throughout antiquity and even up to the 19th and 20th centuries, when it was studied by such thinkers as Abraham Lincoln and Albert Einstein.

Euclid begins with some definitions, some postulates, and some "common notions," as he calls them. The postulates and common notions are self-evident truths, not requiring proof. Then, using the definitions, postulates, and common notions, along with reason, Euclid deduces a consequence, called the first proposition. From there, he deduces a second proposition and so on, building on his foundation of axioms and definitions.

Some of the terms Euclid defines are quite familiar to us, such as an isosceles triangle and a circle. One unusual definition is number 10, in which Euclid defines a right angle using perpendicular lines rather than degrees. Another interesting definition is number 23, in which he defines parallel lines as being in the same plane but never meeting. Euclid does not say that such lines must be everywhere equidistant or have the same slope.

Euclid next defines five postulates: (1) It is possible to draw a straight line from any point to any point. (2) It is possible to produce a finite straight line continuously in a straight line. (3) It is possible to describe a circle with any center in any radius. (4) All right angles are equal. (5) When a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles. A picture helps us visualize this last postulate. Following the postulates, Euclid states his five common notions, which deal with equalities and inequalities.

Euclid's first proof, called proposition I.1, is that an equilateral triangle can be constructed on a finite straight line. This proposition can be demonstrated graphically, using the postulates and common notions. Proposition I.4 is called the side-angle-side congruence scheme; it's a way to show that triangles are congruent. Proposition I.5 proves that the base angles in isosceles triangles are equal. Proposition I.8 is the side-side-side congruence scheme: If the three sides of one triangle equal, respectively, the three sides of another, then the triangles are congruent. Proposition I.11 shows how to draw a straight line at right angles to a given straight line from a given point on it. Again, Euclid shows how to construct the perpendicular line using only the definitions, postulates, and earlier propositions. Finally, proposition I.20 is the triangle inequality; it states: In any triangle, two sides taken together in any manner are greater than the remaining side. Epicurean philosophers later questioned the need to prove this proposition, but a Greek commentator named Proclus defended Euclid: "Granting that the theorem is evident to sense-perception, it is still not clear for scientific thought. Many things have this character; for example, that fire warms. This is clear to perception, but it is the task of science to find out how it warms." ■

Suggested Reading

Euclid, *Euclid's Elements*.

Heath, *A History of Greek Mathematics*.

Loomis, *The Pythagorean Proposition*.

Proclus, *A Commentary on the First Book of Euclid's Elements*.

Questions to Consider

1. Get a copy of Euclid's *Elements*, Book I, proposition 5 (or find it online at http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI5.html). This is his proof that the base angles of an isosceles triangle

are congruent. It immediately follows I.4, which is the side-angle-side congruence scheme (SAS). Read the proof, master it, and thereby cross over the *pons asinorum*!

2. Which side do you support in the Euclid/Epicurean controversy (proposition I.20)? Was Euclid right (as Proclus contended) in *proving* the triangle inequality, even though it is "evident to an ass," or could Euclid have glossed over such trivialities? (Related question: Is the "trivial" really trivial?)

Euclid's *Elements*—Triangles and Polygons

Lecture 4

Thousands of years later at the end of the 18th century … Carl Friedrich Gauss looks back into Euclid and finds something that Euclid had missed. When we get to that—many, many lectures down the road—we will be constructing regular polygons again. This is one of those things [where] "what's old is new again."

We continue our journey through Euclid with proposition I.26, proving the remaining congruence schemes: angle-side-angle and angle-angle-side. These give Euclid a full complement of congruence schemes, which constitute a critical feature of his geometry. In proposition I.27, Euclid introduces the concept of parallels for the first time. According to this proposition, if alternate interior angles are congruent, then the lines are parallel. In I.29, Euclid proves the converse—if two lines are parallel, the alternate interior angles are equal—using postulate 5 (the parallel postulate) for the first time. For the remainder of Book I, he uses this postulate in every proposition, except I.31. His proof of I.29 is done by contradiction.

A few propositions later, in I.32, Euclid uses I.29 to prove an important result: The angles of a triangle must sum to two right angles. As we saw earlier, Thales supposedly completed this proof long before Euclid. Proposition I.46 is also important; this shows how to construct a square on a finite straight line. As you recall, proposition I.1 showed how to construct an equilateral triangle on a line segment, but Euclid couldn't prove the same result for squares until he had established the propositions for parallel lines. Propositions I.47 and I.48 are the highpoints of Book I. According to I.47, in right-angle triangles, the square on the side subtending the right angle is equal to the squares on the sides containing the right angle. You might not recognize it, but that's the Pythagorean theorem. Euclid's graphic proof of this proposition is shown in his "windmill diagram." Proposition I.48 proves the converse: A triangle in which the square on one side is the sum of the squares on the other two sides is a right triangle.

Book III of the *Elements* deals with circles; the first proposition here shows how to find the center of a given circle. Book IV is about constructing regular polygons. Note that a polygon is regular if all the angles are equal and all the sides are equal. Proposition IV.11, for example, shows how to construct a regular pentagon; perhaps the most impressive proposition in Book IV is the last one, in which Euclid shows how to construct a regular pentadecagon, a 15-sided polygon.

In Book VI, Euclid turns to similar figures, which are those that have the same shape but not necessarily the same size. In proposition VI.4, Euclid proves that in triangles with equal corresponding angles, the corresponding sides are proportional. If you've studied trigonometry, you know that it is based almost entirely on the Pythagorean theorem and similar triangles. Proposition VI.8 states: In a right triangle, the altitude to the hypotenuse creates two smaller right triangles, similar to each other and to the original triangle. We'll see this famous result again later in the course.

If you've studied trigonometry, you know that it is based almost entirely on the Pythagorean theorem and similar triangles.

In the next lecture, we'll turn to Book VII, in which Euclid takes up a completely different topic: the phenomenal subject of number theory. ■

Suggested Reading

Euclid, *Euclid's Elements*.

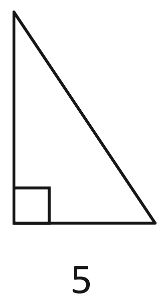
Heath, *A History of Greek Mathematics*.

Proclus, *A Commentary on the First Book of Euclid's Elements*.

Questions to Consider

1. Read and understand proposition I.47, Euclid's famous "windmill" proof of the Pythagorean theorem. This can be found, for instance, at http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI47.html. Euclid's is not the simplest proof of the Pythagorean theorem, but it is surely one of the classic arguments in the history of mathematics.

2. If a right triangle (as shown) has base 5 feet and area 10 square feet, find its perimeter.



Number Theory in Euclid

Lecture 5

People that like math will look at this proof of the infinitude of primes, done in this sort of indirect fashion, and be just enthralled. They are the people that should become mathematicians.

Many people are unaware that Euclid's *Elements* deals with number theory in addition to geometry. Number theory is the study of the properties of whole numbers. It may seem as if these are the simplest sort of mathematical entities, but in fact, number theory is one of the most difficult mathematical subjects.

Euclid begins, again, with definitions, the first of which reads as follows: "A unit is that by virtue of which each of the things that exists is called 1." In other words, the number 1 is a unit. According to the second definition, a number is a multitude composed of units.

In proposition 2 of Book VII, Euclid proves the Euclidean algorithm, which is a method for finding the greatest common divisor of two numbers. He also defines a prime number, which is a whole number greater than 1 that is divisible only by 1 and itself, and a composite number, that is, a whole number greater than 1 that is divisible by some number strictly between 1 and itself (e.g., 15 is composite because it is 3×5, and 3 is intermediate between 1 and 15).

Using the definitions of prime and composite, Euclid proves proposition VII.31: Any composite number has a prime divisor. The number 30, for example, is the product of 10, which is not a prime number, and 3, which is prime. The number 120 is the product of 10 and 12, neither of which is prime, but 10 can be broken down into 2×5, and 2 is prime. Euclid's proof of this proposition for any composite number is a classic result of number theory. His argument showed that any given number can be broken down into a finite chain of intermediate divisors in which each divisor is greater than 1 and smaller than its predecessor. The chain stops when we reach a prime divisor.

In Book IX, Euclid proves that there are infinitely many primes. An easy way to prove infinitude of some entity is to identify a pattern and just spin out results, as we see with the example of the squbes. But this method wasn't available to Euclid because there is no pattern for generating primes. In proposition 20 of Book IX, Euclid stated his theorem about primes as follows: "Prime numbers are more than any assigned multitude of prime numbers." In other words, no finite collection of primes contains all the prime numbers. If we start with a finite collection of primes—*a*, *b*, *c*, *d*, *e*—and we define *n* as the product of these primes + 1, our results will fall into one of two cases: (1) *n* itself may be a new prime number or (2) *n* will be composite and will have a prime divisor (*p*) that is not among the original primes.

Any given number can be broken down into a finite chain of intermediate divisors in which each divisor is greater than 1 and smaller than its predecessor. The chain stops when we reach a prime divisor.

Book X of the *Elements* is about something called quadratic surds; Books XI and XII venture into the realm of solid geometry; and in Book XIII, Euclid proves that there are only five regular solids (also called Platonic solids): the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. With that proof, the *Elements* comes to an end. Ivor Thomas, a math historian, likened Euclid's work to the perfection of the Parthenon. The *Elements* is a great legacy of the Greeks' affinity for beauty and symmetry. ■

Suggested Reading

Euclid, *Euclid's Elements*.

Hardy, *A Mathematician's Apology*.

Heath, *A History of Greek Mathematics*.

Ore, *Number Theory and Its History*.

Weil, *Number Theory*.

Questions to Consider

1. (a) True or false: There are infinitely many *odd* **prime** numbers.

(b) True or false: There are infinitely many *odd* **composite** numbers.

2. Revisit Euclid's proposition IX.20, in which he proved that no finite collection of primes can contain *all* the primes. Start with primes a = 2, b = 5, c = 7, and d = 11 and form the number $N = (a \cdot b \cdot c \cdot d) + 1$. Show that N is not prime, but show as well that its prime divisors are "new" in the sense that they are not 2, 5, 7, or 11. This is exactly what Euclid proved in case 2 of his wonderful argument.

The Life and Works of Archimedes

Lecture 6

"These properties were all along naturally inherent in the figures referred to … but remained unknown to those who were before my time engaged in the study of geometry." –Archimedes.

Archimedes was born in 287 B.C. in Syracuse, a city on the island of Sicily. He apparently lived most of his life in Syracuse, although there is some evidence that he might have studied at Alexandria. Much of our knowledge of Archimedes comes from Plutarch's *Life of Marcellus*, in which the mathematician figures as a supporting character. According to Plutarch, Archimedes was so engulfed in his work that he often forgot to eat or bathe.

These properties were always there, but it took Archimedes to see them.

Archimedes is known to some through the story of King Hieron and his crown. Hieron had given his goldsmith some gold to make a crown, but the king was suspicious that the goldsmith might have substituted a lesser alloy for the gold and asked Archimedes to help determine whether this was the case. According to the story, while bathing, Archimedes noticed that the water sloshed out of the tub when he lowered himself into it; he then realized that by lowering the crown into water, he could calculate its density and compare it to the density of gold to see if a substitution had been made. When he hit on this discovery, Archimedes is said to have leapt out of the bath and walked home naked, shouting "Eureka!"

Among the works that have come down to us from Archimedes are *On Floating Bodies*, *On Spirals*, and *Quadrature of the Parabola*, a book concerned with finding the area under a curve. His work entitled *Measurement of a Circle* has only three propositions, two of which are classics. In the first of these, he asserts that the area of any circle is equal to a right-angled triangle in which one of the sides of the right angle is equal to the radius and the other is equal to the circumference of the circle. We'll return to this proposition in the next lecture. In proposition 3 of this work, Archimedes states that the ratio of the circumference of any circle to its diameter is less than 3 1/7 but greater than 3 10/71. Recall that the ratio of the circumference of a circle (the distance around) to the diameter (the distance across) is a constant, π. In his calculations, Archimedes determined π to two decimal places, 3.14, although he expressed it in fractions rather than decimals. He accomplished this proof by approximating the circle's circumference by the perimeters of inscribed and circumscribed regular 96-gons.

Archimedes's masterpiece is *On the Sphere and the Cylinder*, in which he found the volume and surface area of a sphere. According to Archimedes, "If a cylinder is circumscribed about a sphere, the cylinder is half as large again as the sphere in volume and half as large again as the sphere in surface area." We can generate formulas for these amazing results using modern-day notation, and their accuracy can be confirmed through calculus.

Archimedes died during the siege of Syracuse by the Romans in 212 B.C. He had been given the job of defending the walled city against land and sea attacks ordered by the general Marcellus. Archimedes devised devilish weapons of war for Syracuse, against which the Romans were initially defenseless. When the Romans finally broke through the walls, however, Archimedes was found and commanded to appear before Marcellus. Because he was working on a proof at the time of his capture, Archimedes declined to follow his captor and was immediately slain. His memorial, according to Cicero, was a small column topped by a sphere contained in a cylinder. ■

Suggested Reading

Boyer, *The Concepts of the Calculus*.

Edwards, *The Historical Development of the Calculus*.

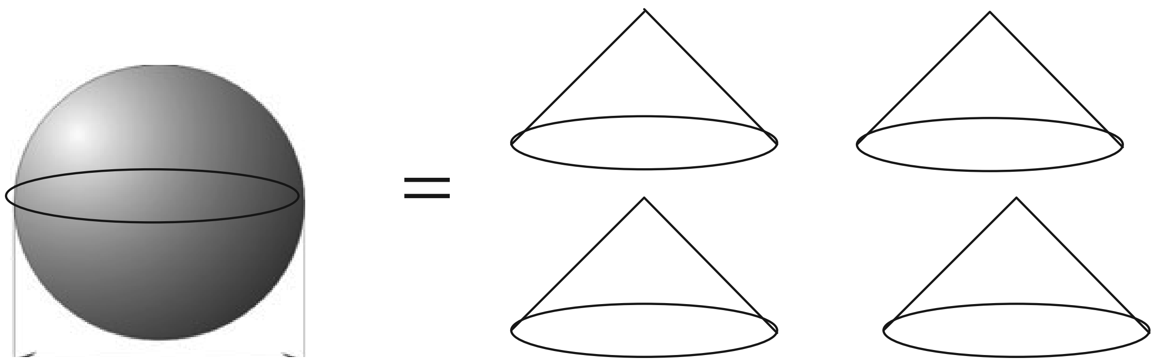
Heath, *A History of Greek Mathematics*.

Questions to Consider

1. A letter has survived from Archimedes to Eratosthenes, who was the librarian at Alexandria. Look up Eratosthenes, find out about his "sieve" for finding prime numbers, and read how he determined the size of the

Earth using only a few simple astronomical observations and Euclid's proposition I.29.

2. In proposition 34 of *On the Sphere and the Cylinder*, Archimedes stated, "Any sphere is equal [in volume] to four times the cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere." Show that this yields the correct formula for spherical volume. Archimedes (of course) was right again!



Archimedes's Determination of Circular Area
Lecture 7

A modern reader looks at this and says this is strangely indirect. What a strange way to prove that two things are equal: to show that one can't be bigger than the other or smaller than the other; hence, they must be equal.

Recall from the last lecture the first proposition of Archimedes in *Measurement of a Circle*: "The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius and the other to the circumference of the circle." What he's really saying is that a circular area is the same as that of a particular triangle.

The Greeks often used the strategy of comparing a complicated figure to a simpler one. In this case, the circle is the more complicated figure and the triangle is the simpler. To find the area of a triangle, we return to Euclid's *Elements*. Proposition I.41 states, ""If a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle." Unpacking this statement, we find the formula 1/2(bh) for the area of a triangle. In Archimedes's triangle, the base is c, the circumference of the circle, and the height is r, the radius; thus, the formula for the area is 1/2(cr).

Archimedes had to establish two preliminary results, often called lemmas in mathematics. Think of a lemma as something we prove because we will need it later.

To proceed further, Archimedes had to establish two preliminary results, often called lemmas in mathematics. Think of a lemma as something we prove because we will need it later. First, Archimedes wanted to determine the area of a regular polygon based on a measurement called the apothem (the perpendicular distance from the center of the polygon to any of the sides) and the perimeter. The formula for this is 1/2 the product of the perimeter and the apothem. The second lemma needed can be stated as follows: Given a circle, we can inscribe within it a square, then double the number of sides and get a regular octagon, then double the number of sides and get a regular 16-gon, and continue until the difference between the circle's area and that of the inscribed polygon is as small as we wish. This approach is sometimes called the method of exhaustion. In modern language, we would say that the limit of the areas of these polygons approaches the area of the circle. Note, however, that there is no polygon whose area is exactly the area of the circle from the inside because polygons have straight sides, while the circle is round.

With these lemmas, Archimedes was able to prove the original proposition using a method we might call double *reductio ad absurdum* or a double contradiction. He first assumes that the area of the circle is greater than the area of the triangle. Using the two lemmas, he finds, first, that the area of a polygon inscribed within the circle is greater than that of the triangle; then he finds that this same area is less than that of the triangle. Having reached a contradiction, he eliminates this first assumption; he then uses a similar process to eliminate the assumption that the area of the circle is greater than that of the triangle. Finally, Archimedes states that if the area of the circle is neither greater nor less than the area of the triangle, the two are equal. We can trace Archimedes's proof to find our familiar formula for the area of a circle: πr^2 . We can also take a more direct approach to determining the area of a circle using integral calculus, which simplifies the process into a series of steps and allows us all to become mathematicians like Archimedes. ■

Suggested Reading

Boyer, *The Concepts of the Calculus*.

Edwards, *The Historical Development of the Calculus*.

Heath, *A History of Greek Mathematics*.

Questions to Consider

1. We saw that the area of a regular polygon is half the product of its perimeter and its apothem. As a formula, this becomes: area = $\frac{1}{2}Ph$.

Suppose our regular polygon is a *square*. Show that our formula yields the correct area in this (simple) case.

2. As we saw, Archimedes established his formula for circular area by the logical strategy of double *reductio ad absurdum* (double contradiction). Conjure up a situation—it need not have to do with mathematics—in which a result could be established by **triple** *reductio ad absurdum*.

Heron's Formula for Triangular Area

Lecture 8

I challenged you to find the square root of 336. … I'll give you nothing but a stack of paper and a pencil or maybe a stack of papyrus if we want to be in the spirit of Heron. How do you do something like this?

Our last Greek mathematician is Heron of Alexandria (sometimes called Hero of Alexandria), who is usually dated to around the year 75. Heron is credited with inventing the aeolipile, a proto-steam engine, and with devising a method for drilling a tunnel from both sides of a mountain at once. His great work was the *Metrica*, in which we find his method for approximating square roots and his formula for triangular area.

Suppose we want to find $\sqrt{336}$, with Heron's method, we begin with an estimate (x_1). We'll try 18; 18^2 is 324. For a second estimate (x_2), we take x_1 divided by 2 + the number whose square root we seek divided by 2 × x_1; in this case, the expression would read 18/2 + 336/2(18), which yields 55/3, or 18 1/3. We then repeat the process to get a third estimate, which yields 6049/330, or 18 109/330. If we square this result, we get very close to 336. The modern idea of limits shows us why this technique works. The general idea is as follows: Let x_n be the approximation at any stage of $\sqrt{A}$. The next approximation, $x_n + 1$, will be $x_n/2 + A/2x_n$. As n goes to infinity, we see that x_n approaches $\sqrt{A}$.

An even greater result in the *Metrica* is the formula for finding triangular area without knowing the altitude of the triangle. For a triangle with three sides, a, b, and c, we first find the semiperimeter, s, which is half of the perimeter. Heron's formula for the area of the triangle then is $\sqrt{s(s-a)(s-b)(s-c)}$. This formula seems implausible because it is completely unrelated to the formula we usually use, 1/2(bh), but if we substitute values for a, b, and c, we can see that it works.

Suppose we had a four-sided plot of land measuring 10 × 17 × 25 × 24 yards. There is no unique answer for the area of this quadrilateral. Just knowing the side measurements of a four-sided figure does not determine the area, but we can find the area of a three-sided figure using the side measurements. Thus, if we draw a diagonal in the quadrilateral, we can find its area by finding the area of the two resulting triangles. In this case, say the diagonal measures 26 yards. Using Heron's formula, we determine the area of one triangle to be 120 and the second triangle to be 204; thus, the area of the four-sided plot of land is 324 square yards. In fact, we can break any polygon into triangles, apply Heron's formula repeatedly, and sum the results to find the total area.

Heron's proof of this formula is quite complicated. As we will see in a later lecture, Isaac Newton offered a much simpler proof. Interestingly, Heron's formula implies the Pythagorean theorem as a consequence, which we can see by finding the area of two congruent triangles using the usual formula and Heron's formula.

Heron's proof of this formula is quite complicated. As we will see in a later lecture, Isaac Newton offered a much simpler proof.

The Greeks were impressive in their ability to delve deeply into mathematics with limited tools and without many of the modern mathematical results that we take for granted. With the work of Thales, Pythagoras, Euclid, Archimedes, and Heron, the Greek mathematical legacy is unsurpassed. ■

Suggested Reading

Heath, *A History of Greek Mathematics*.

Questions to Consider

1. Starting with a first approximation of $x_1 = 10$, use two applications of Heron's technique to approximate $\sqrt{105}$. How accurate is the estimate?

2. Use Heron's formula to find the area of a triangle whose three sides are of length 25, 52, and 63.

Al-Khwārizmī and Islamic Mathematics

Lecture 9

[The quadratic formula] is the most famous and most useful formula in algebra. This will solve any quadratic equation for you.

The golden age of Islamic science, scholarship, and mathematics is usually set between the 8th and 13th centuries. During this time, Islam had grown from its origins and spread across North Africa into Spain and Sicily, then spread eastward to India. In the heart of this glorious civilization, Baghdad, was an intellectual center called the House of Wisdom. Islamic mathematicians at this time took three important directions in their work. First, they studied and translated the Greek texts, such as Euclid's *Elements* and the work of Ptolemy known to us as *The Almagest*. They also absorbed Indian mathematics, which included trigonometry, arithmetic, and the base-10 numeral system that we still use today. One example of Indian mathematics from the 7th century is Brahmagupta's formula for the area of a cyclic quadrilateral, which yields Heron's formula for triangular area that we saw in the last lecture. Finally, Islamic mathematicians were known in particular for their advances in solving equations, and the name that comes up most often in this regard is Muhammad Mūsā ibn Al-Khwārizmī (c. 780–850).

Al-Khwārizmī's important book was titled, in Arabic, *Hisab al-jabr w'al-muqābala*. The word *al-jabr* later became Latinized to "algebra," and Al-Khwārizmī's name itself was Latinized to "algorithm."

Al-Khwārizmī's origins are unclear, but he eventually gravitated to the House of Wisdom in Baghdad and became a major scholar there, working in mathematics, geography, astronomy, and astrology. He wrote two important books, one of which was *On the Calculation with Hindu Numerals*. This work was translated into Latin around the year 1200 and served as the introduction of Hindu-Arabic numerals into Europe. Al-Khwārizmī's other important book was titled, in Arabic, *Hisab al-jabr w'al-muqābala*. The word *al-jabr* later became Latinized to “algebra,” and Al-Khwārizmī’s name itself was Latinized to “algorithm.”

Through a process that he called restoring and comparing, Al-Khwārizmī solved first-degree linear equations and second-degree quadratic equations. He did this by transferring the equations into one of six forms. Al-Khwārizmī used six forms rather than the two we would use because negative numbers were not considered legitimate. For instance, we could solve the quadratic equation $3x^2 - 5x = -2$, but if negative numbers were not allowed, we would have to rewrite the equation as $3x^2 + 2 = 5x$. We could then refer to Al-Khwārizmī’s chapter on squares and numbers equal to roots to find a solution. We see an example of Al-Khwārizmī in action while solving the following problem: “What must be the square which, when increased by 10 of its own roots, amounts to 39?” In modern notation, the problem is: $x^2 + 10x = 39$. Looking at this problem geometrically helps us understand Al-Khwārizmī’s solution process and arrive at the result, which is 3. This technique is now referred to as completing the square. We can also apply this process to the generic quadratic equation. Here, we do algebraically what Al-Khwārizmī did geometrically, and we end up with the quadratic formula: $x = -b \pm \sqrt{b^2 - 4ac}\,/2a$.

Other great Islamic mathematicians who came after Al-Khwārizmī include the number theorist Thābit ibn Qurra and the writer of the *Rubaiyat*, Omar Khayyam, who developed a geometric technique that proved useful for solving cubic equations. Islamic works eventually found their way back to Europe and helped to revive European mathematical learning in the 12th century, at the dawn of the Renaissance. We’ll visit this period in the next lecture. ■

Suggested Reading

Joseph, *The Crest of the Peacock*.

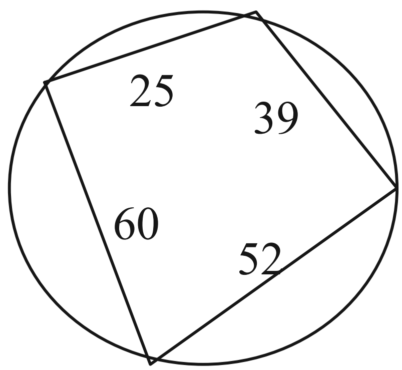
Katz, ed., *The Mathematics of Egypt, Mesopotamia, China, India, and Islam*.

Plofker, *Mathematics in India*.

Questions to Consider

1. Use Bhramagupta's formula to find the area of the cyclic quadrilateral shown.

2. Use al-Khwārizmī's recipe to find the number for which (in archaic language) "the square plus six of its roots is 40." In other words, complete the square to solve the quadratic equation $x^2 + 6x = 40$.



A Horatio Algebra Story

Lecture 10

It has been said that the 16th century produced three great scientific works. … We have a great work in medicine [Vesalius's treatise on the human body], we have a great work in astronomy [Copernicus's *De revolutionibus*], and the third major scientific work from the century is Cardano's *Ars Magna* of 1545.

By the later 15th century, the Renaissance was up and running in Europe. The early universities were thriving at Padua and Bologna; with the advent of the printing press in 1450, books were widely available; and in 1492, Columbus discovered the New World. Europeans had absorbed the learning of the past and were now starting to push the frontiers of knowledge.

Our story in this lecture begins in 1494, when Luca Pacioli published the *Summa de Arithmetica*. This book described how to solve first- and second-degree equations but stated that a solution to the cubic equation (written in modern form as $ax^3 + bx^2 + cx + d = 0$) was impossible. The Europeans sought a "solution by radicals," defined as a formula that gives an *exact* solution and uses only the coefficients of the cubic equation and the "algebraic" operations of addition, subtraction, multiplication, division, and root extraction. Such a solution would be analogous to the quadratic formula.

Amazingly, Tartaglia independently discovered the solution of the depressed cubic and was able to solve all the problems given him by Fiore.

Early in the 16th century, a mathematician named Scipione del Ferro found a solution for the depressed cubic, a somewhat more restricted form than the general cubic. This equation is written as $x^3 + mx = n$; notice that the x^2 term is missing. On his deathbed, del Ferro passed this solution on to one of his students, Antonio Fiore. Fiore, in turn, used del Ferro's legacy to challenge the greatest mathematician in Italy at the time, Niccolo Fontana, nicknamed Tartaglia, the

"Stammerer." Amazingly, Tartaglia independently discovered the solution of the depressed cubic and was able to solve all the problems given him by Fiore.

At this point, a fellow named Gerolamo Cardano heard about the mathematical challenge and sought out Tartaglia to learn the solution to the depressed cubic. Cardano was a strange character, as we know from his autobiography, *De Vita Propria Liber*. He regularly conversed with his guardian angel and was a serious gambler but also became one of Europe's foremost physicians and wrote more than 100 books on a wide array of subjects. In 1539, Cardano promised Tartaglia that he would keep the solution to the depressed cubic a secret.

At around this time, Cardano began teaching mathematics to a brilliant young man named Ludovico Ferrari. In the course of their work together, Cardano discovered the solution to the general cubic equation, and his protégé discovered the solution to the quartic equation (written as $ax^4 + bx^3 + cx^2 + dx + e = 0$). But both of these solutions rested on the ability to solve the depressed cubic, which Cardano had promised Tartaglia he would not reveal. In 1543, Cardano went back to del Ferro's original papers and used those as the basis for publishing his great algebraic discoveries in the treatise *Ars Magna*.

Of course, Tartaglia was enraged and challenged Cardano to a mathematical contest. Cardano refused the challenge, but his protégé, Ferrari, accepted on his mentor's behalf. In the end, Ferrari bested Tartaglia, who disappeared into history, while the glory of this mathematical discovery is still attributed to Cardano and the *Ars Magna*. ■

Suggested Reading

Cardan, *The Book of My Life*.

Hald, *A History of Probability and Statistics and Their Applications before 1750*.

Ore, *Cardano*.

Questions to Consider

1. The secret to solving a general third-degree equation rests in the special case of the "depressed cubic." This is a third-degree equation lacking its next-highest degree (i.e., second-degree) term—what Cardano would describe as "cube plus roots equals number" or we would write as $x^3 + mx = n$. Using this terminology, identify what a "depressed quadratic" would look like and indicate why it would be easy to solve.

2. We noted that Cardano's *Ars Magna* (1545) is placed alongside Vesalius's *De humani corporis fabrica* (1543) and Copernicus's *De revolutionibus* (1543) as the supreme scientific works of the 16th century. Why are the other two books so highly esteemed as scientific milestones?

To the Cubic and Beyond

Lecture 11

You can sort of see at this point why they don't teach [the depressed cubic formula] in schools. It's hard enough to remember the quadratic formula, let alone this one.

In this lecture, we'll see how to solve the cubic equation, but we'll warm up by looking at a second-degree equation. As we've seen, the quadratic formula allows us to solve any second-degree equation of the form $ax^2 + bx + c = 0$. All we have to do is substitute the coefficients a, b, and c into the formula $x = -b \pm \sqrt{b^2 - 4ac}\,/2a$. This is an analog of what we're seeking for the third-degree equation.

To solve the cubic equation, we will follow Cardano in *Ars Magna*. We begin with the depressed cubic, written as $x^3 + mx = n$, but we'll substitute numbers into the equation: $x^3 + 24x = 56$. Cardano looked at this problem in much the same way that Al-Khwārizmī looked at the second-degree equation. Instead of completing the square, however, Cardano approached the problem by subdividing a cube. He then had to find the volume of the cube by adding up the pieces. When we go through this process, we're left with the expression $u^3 + (t - u)^3 + (t - u)[2tu + u^2 + u(t - u)]$, in which t represents the height, length, and depth of the cube and u represents units. We set this expression equal to t^3, the volume of a cube. Simplifying, we get $(t - u)^3 + [3tu](t - u) = t^3 - u^3$, which has the same structure as the equation we're trying to solve, $x^3 + 24x = 56$. We substitute x for $t - u$, and Cardano's equation becomes $x^3 + [3tu]x = t^3 - u^3$. We now designate the coefficient $3tu$ to be 24 and $t^3 - u^3$ to be 56.

This leaves us with two equations in two unknowns, t and u. We approach these by solving one equation for one variable and substituting that back into the other equation. Here, we end up with $t^3 - 512/t^3 = 56$, leaving us with one equation and one unknown, t. To eliminate the t^3 in the denominator, we multiply both sides by t^3, which yields $t^6 - 56(t^3) - 512 = 0$. This is a six-degree equation in t, but it is second degree in t^3. It can be written as $t^6 - 56(t)^2 - 512 = 0$, which looks just like a quadratic equation. If we let y

play the role of t^3, we get the exact second-degree equation we saw at the beginning of this lecture: $y^2 - 56y - 512 = 0$. For y, or t^3, we then choose one of the two solutions we found earlier, 64 or −8, and work our way back up the chain to a solution. Using 64, we find that the solution to our earlier cubic, $x^3 + 24x = 56$, works out to be $x = t - u = 4 - 2 = 2$.

For the generic depressed cubic, the formula is: $\sqrt[3]{\frac{n}{2}+\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}}-\sqrt[3]{\frac{n}{2}\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}}$. We can try this formula with the example done in the lecture, substituting 24 for m and 56 for n. What if the cubic isn't depressed, as in the generic $ax^3 + bx^2 + cx + d = 0$? Cardano discovered that he could make the equation depressed by replacing x with $y - b/3a$, where the a's and b's are the coefficients of the general cubic.

It has since been shown that a solution by radicals of the general quintic or any higher-degree equation is impossible.

Cardano's *Ars Magna* also includes Ferrari's solution of the quartic, which he accomplished by reducing the fourth-degree equation to a third degree, then to a depressed cubic. It has since been shown that a solution by radicals of the general quintic or any higher-degree equation is impossible. ■

Suggested Reading

Cardano, *Ars Magna*.

Ore, *Cardano*.

Questions to Consider

1. Use Cardano's formula to get a solution by radicals for the depressed cubic $x^3 + 15x = 124$. Check your answer.

2. Solve the cubic equation $x^3 + 9x^2 + 42x = 52$ by first making the substitution $x = y - 3$ to reduce this to a depressed cubic in y. HINT: The depressed cubic you get should look familiar.

The Heroic Century

Lecture 12

Descartes' writing is regarded as being particularly opaque, particularly hard to follow. He himself seemed to take pride in this for some reason. … Descartes argued that only by making the reader struggle would the reader truly learn the material. It was his goal, his job, to make it hard to follow.

The 17th century is sometimes called the heroic century in mathematics because this period saw the development of many ideas that are now fundamental to our understanding of mathematics and many that are still being studied and researched.

Our first figure from this period is François Viète, who gave us a more modern algebraic notation in his book *The Analytic Art* (1591). As we've seen, earlier mathematicians had to express their equations in long, complicated sentences. Viète introduced the practice of allowing letters to stand for what had been termed roots, squares, and cubes.

Pierre de Fermat is perhaps the greatest mathematician from the heroic century.

Logarithms were another great innovation from this time, discovered and developed by John Napier and Henry Briggs. Together, these men created what are called the common, or Briggsian, logarithms. For example, the $\log_{10}$ of 100 is 2, because 100 is 10^2; the $\log_{10}$ of 1000 is 3 because 1000 is 10^3. In 1624, Briggs published *Arithmetica Logarithmica*, which was a table of logarithms that he had painstakingly calculated to 14-place accuracy.

Another significant achievement came from the well-known philosopher René Descartes. In an appendix to his *Discourse on Method* (1637), Descartes gave us the first published account of analytic geometry, that is, the idea of applying algebra to geometric figures in the plane. He also gave us modern algebraic notation, as we see in his expression of the depressed cubic.

Another Frenchman, Blaise Pascal, invented a mechanical calculating machine in 1642. Pascal was also one of the first people to transform probability into a mathematical science. One of his specific interests was the quadrature of the cycloid. The quadrature is the area under a curve, and a cycloid is a type of curve traced by a point on a moving wheel. Perhaps this mathematician is most well known for the development of Pascal's triangle, an array of numbers that is used in expanding binomials. For example, if we wanted to cube the binomial $a + b$, we could find the coefficients in row 3 of Pascal's triangle (1, 3, 3, 1) and, thus, get the expansion: $a^3 + 3a^2b + 3ab^2 + b^3$.

Pierre de Fermat is perhaps the greatest mathematician from the heroic century. Fermat corresponded with Pascal on the development of probability theory, created his own version of analytic geometry, and foreshadowed both differential and integral calculus. His primary achievement was his work in number theory. The Fermat factorization scheme, for example, is simple but very clever. According to this, if we want to factor n, we let a be the largest whole number that is greater than or equal to $\sqrt{n}$. Then, we look at differences: $a^2 - n$, $(a + 1)^2 - n$, $(a + 2)^2 - n$, and so on. At some point, when we take the square of a number and subtract n, we will get a perfect square. We can then rearrange the problem into a difference of squares and factor it algebraically; the results will provide the factorization of n.

One other result of Fermat is motivated by the fact that sometimes two squares sum to a square; for example, $3^2 + 4^2 = 5^2$. According to Fermat, this is impossible with higher powers. In his notes, Fermat wrote that he had a proof of this assertion, but the margin in which he was writing was "too narrow to contain it." In fact, it wasn't until 1995 that this result, known as Fermat's last theorem, was finally proved—and the proof required hundreds of pages! ■

Suggested Reading

Boyer, *A History of Analytic Geometry*.

Descartes, *The Geometry of René Descartes*.

Devlin, *The Unfinished Game*.

Hald, *A History of Probability and Statistics and Their Applications before 1750*.

Ore, *Number Theory and Its History*.

Weil, *Number Theory*.

Questions to Consider

1. Use Fermat's factorization scheme to factor 2,373,793 into the product of two smaller numbers. Start with the fact that $\sqrt{2{,}373{,}793}$ falls between 1540 and 1541. NOTE: Thanks to Fermat's insight, this is much easier than it looks.

2. Another of Fermat's number theoretic results is called "the little Fermat theorem." This says that if *p* is a prime and *a* is any whole number, then *p* divides evenly into $a^p - a$.

(a) Check this numerically for the (easy) example of $p = 3$ and $a = 5$.

(b) Now check it for the (considerably less easy) example of $p = 17$ and $a = 2$.

The Legacy of Newton

Lecture 13

It's been said that Newton's *Principia* is the greatest science book ever written, and that might well be true. If it has any rival, it would be Darwin's *Origin of Species* from centuries later.

Isaac Newton was born in 1642 to a widowed mother in a small town in Lincolnshire. As a child, he was said to have been sober, silent, and very smart. In 1661, he entered Trinity College, where he was mentored by Isaac Barrow, the Lucasian Professor of Mathematics. Barrow directed Newton's readings into Descartes and other mathematical thinkers.

The years 1665–1667 were the *anni mirabilis*—miraculous years—for Newton, the time when he pushed the frontiers of mathematics and science. In this period, he discovered the generalized binomial theorem, differential and integral calculus, and the laws of motion—all while a student at Trinity College. When the college had to shut down because of an outbreak of the plague, Newton went home, where he developed the theory of universal gravitation. According to contemporary descriptions of Newton, he became so caught up in his work that he often neglected to eat or sleep and never engaged in any outside recreation or pastime. Nor was Newton in any hurry to share his discoveries with others. He believed that publication would bring with it an increase in social activities, which he desired to avoid.

In 1669, he was appointed to succeed Barrow as Lucasian Professor of Mathematics. At about the same time, he invented the reflecting telescope, which he sent to the Royal Society in London in 1671. As a result, Newton was made a member. The next year, he submitted some of his papers on optics to the Royal Society but received criticism from the scientist Robert Hooke. For much of the remainder of the 1670s and 1680s, Newton abandoned mathematics and science to devote his time to alchemy and theology.

In the later 1680s, with the urging and financial backing of Edmund Halley, Newton finally began to publish some of his work in mechanics. In 1687, Newton's great work the *Principia Mathematica* appeared. Here, he described the laws of motion and gravity in a very Euclidean fashion, putting forth definitions and axioms, then deducing propositions. The *Principia* launched Newton into fame, whether he wanted it or not.

In 1689, Newton was elected to Parliament, but his government career was brief and not particularly memorable. In the years 1692–1694, he seems to have suffered a period of mental derangement, perhaps brought on by mercury poisoning in connection with his work on alchemy. Fortunately, he recovered and, in 1696, became warden of the mint. He apparently was quite successful in this position, overseeing the recoinage in Britain and prosecuting counterfeiters with great zeal.

In 1687, Newton's great work the *Principia Mathematica* appeared. Here, he described the laws of motion and gravity in a very Euclidean fashion, putting forth definitions and axioms, then deducing propositions.

In 1703, Newton was made president of the Royal Society, a position he held for the rest of his life. In 1704, he published his *Opticks*, a work on light and color. In 1705, he was knighted by Queen Anne. By this point, Newton was quite famous and wealthy, but he remained crotchety and contentious throughout his life. His scholarly disputes included those with Robert Hooke; John Flamsteed, the royal astronomer; and Gottfried Wilhelm Leibniz, the German mathematician and philosopher who invented calculus independently of Newton and at around the same time.

Newton died in 1727 and was buried in Westminster Abbey. Voltaire reported that the British "buried him as though he had been a king …," and Wordsworth later described the statue of him that stands in Trinity College chapel: "Of Newton with his prism and silent face, / The marble index of a mind for ever / Voyaging through strange seas of Thought, alone." ■

Suggested Reading

Fauvel, et al., *Let Newton Be!*

Gjertsen, *The Newton Handbook*.

Hall, *Philosophers at War*.

Newton, *The Correspondence of Isaac Newton*.

Westfall, *The Life of Isaac Newton*.

———, *Never at Rest: A Biography of Isaac Newton*.

Questions to Consider

1. To get a sense of the era, read about mathematician Isaac Barrow (Newton's mentor), physicist Robert Hooke (Newton's nemesis), and counterfeiter William Chaloner (Newton's prisoner).

2. Newton lived during a turbulent time in the history of England. He was born in 1642, at the start of the English Civil War, which culminated, in 1649, with the execution of King Charles I; he began his Cambridge career the year after the monarchy was restored under King Charles II; and by mid-career, Newton saw King James II flee the country for reasons of religion. Read up on these times, and imagine how a person like Newton would have reacted to all the turmoil.

Newton's Infinite Series

Lecture 14

Newton's insight in 1664 was that you could use this expansion not just for whole numbers *r* but for other kinds of exponents. What he said is, *r* could be integral or (so to speak) fractional, positive or negative. … Nobody had ever thought of that.

The exploration of infinite series was at the cutting edge of research in 17th-century mathematics, and Newton was at the forefront of this study. In this context, "series" means sum, and an infinite series is the sum of infinitely many terms.

We begin with a finite series. We could use the FOIL method to solve the binomial $(1 + x)^2$, but for a problem like $(1 + x)^{17}$, we need something called the binomial expansion. Let's consider $(1 + x)^r$. The expansion starts with 1. When we multiply $1 + x$ by itself r times, all the 1's will multiply. We then get $r \times x$. For the third term, we get $r(r - 1)/2 \times 1 \times x^2$. The x^3 term has as its coefficient $r(r - 1)(r - 2)/3 \times 2 \times 1$, and the pattern continues. We are always marching down from r: $r - 1$, $r - 2$, and so on.

Newton realized that the binomial expansion could be used even if the exponent was not an integer.

Newton realized that the binomial expansion could be used even if the exponent was not an integer. As we go through the math, keep in mind that we can use the simplified notation 4! for an expression like $4 \times 3 \times 2 \times 1$. Also note that a negative exponent, such as x^{-n}, essentially means $1/x^n$. Likewise, a fractional exponent, such as $x^{p/q}$, means $\left(\sqrt[3]{x}\right)^p$.

Suppose we want to expand $(\sqrt[3]{1} + x)$ as a series. We first turn the cube root into a fractional exponent: $(1 + x)^{1/3}$. When we use the binomial expansion, the role of r will be played by 1/3, as follows: $(1 + rx)$ will become $1/3(x)$. Next comes $1/3(1/3 - 1)/2!(x^2)$, followed by $1/3(1/3 - 1)(1/3 - 2)/3!(x^3)$. Simplifying, we get $1 + 1/3(x) + 1/3(-2/3)/2 \times 1(x^2)$, because $1/3 - 1$ is $-2/3$. The next term will be $1/3 \times (-2/3)(-5/3)/3 \times 2 \times 1\ (x^3)$, because $1/3 - 2$ is

–5/3, and so on. Cleaning up the fractions, we get $1 + 1/3(x) - 1/9(x^2) + 5/81(x^3) - 10/243(x^4)\ldots$. This is an infinite series.

Newton thought this tool was useful for finding roots. Suppose we were asked to find $\sqrt[3]{140}$. When we use Newton's generalized binomial expansion, we must be sure that the value of x we use is smaller than 1 so that the terms zero in on an answer. To find that value, we look for a perfect cube that is close to 140, such as 125, which is 5^3. We rewrite the problem as $\sqrt[3]{125 + 15}$, then factor out 125: $\sqrt[3]{125(1 + 15/125)}$. Next, instead of $\sqrt[3]{125}$ multiplied by the parenthetical, we rewrite the expression as $\sqrt[3]{125}\,(\sqrt[3]{1+3/25})$. We break these apart into two cube roots, and what we're left with is $\sqrt[3]{1+3/25}$. We can now use the binomial expansion because x will be 3/25. If we put that value into the series, it will converge quickly to an answer. Stopping after the first four terms, we get: 1 + 1/3 of x will be a 1/3 of 3/25 minus 1/9 of x^2 will be minus 1/9 of $(3/25)^2$ plus 5/81 of $(3/25)^3$. Collecting terms, we are left with 9736/9375, or 5.19253, as our approximation of $\sqrt[3]{140}$.

Newton also did the infinite series expansion of the sine, a well-known trigonometric ratio. His sine series is: $\sin x = x - x^3/3! + x^5/5! - x^7/7! \ldots$, which has become an important result in mathematics down to the present day. ■

Suggested Reading

Boyer, *The Concepts of the Calculus*.

Dunham, *The Calculus Gallery*.

Edwards, *The Historical Development of the Calculus*.

Gjertsen, *The Newton Handbook*.

Grattan-Guinness, *From the Calculus to Set Theory*.

Westfall, Richard. *Never at Rest*.

Whiteside, ed., *The Mathematical Works of Isaac Newton*.

Questions to Consider

1. Use the first four terms in Newton's generalized binomial expansion to approximate $\sqrt[3]{72}$. Recall the trick: $72 = 64 + 8 = 64\left(1+\frac{8}{64}\right) = 64\left(1+\frac{1}{8}\right)$. Your approximation should be quite accurate.

2. Convert $30°$ to radian measure, then use the first three terms of Newton's series for the sine (i.e., $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$) to approximate the sine of this angle. NOTE: As you might recall from trigonometry, the exact value is $\sin(30°) = 1/2$.

Newton's Proof of Heron's Formula

Lecture 15

The geometry in Newton's argument is much reduced from that of Heron, but there's a price to pay. If the geometry goes down, the algebraic sophistication goes up, and as you'll see, the algebra is where the heavy lifting occurs in Newton's proof. You win some, you lose some.

In this lecture, we look at Newton's proof of the formula by Heron we used earlier to find the area of a triangle and a quadrilateral. As you may recall, for a triangle with three sides, *a*, *b*, and *c*, we find the semiperimeter, *s*, and ultimately, perform the following calculation: $\sqrt{s(s-a)(s-b)(s-c)}$. Heron's original geometric proof of this implausible result was quite complicated; Newton tried an algebraic approach.

Newton's proof appeared in his textbook on algebra, *Arithmetica Universalis*, published in 1707. At this point in his career, Newton was experiencing a renewed interest in geometry and devoted his time to improving proofs of elementary theorems. He challenged himself to find algebraic proofs of Heron's formula and similar geometric results. For example, one problem he tackled was to find the hypotenuse of a right-angled triangle given the area, *a*, and perimeter, *p*. The result? The hypotenuse is $p/2 - 2a/p$.

Newton was experiencing a renewed interest in geometry and devoted his time to improving proofs of elementary theorems. He challenged himself to find algebraic proofs of Heron's formula and similar geometric results.

Newton stated the theorem for Heron's formula as follows: "If from half the collected sum of the sides of a given triangle the sides are individually subtracted, the square root of the continued product of that half and the remainders will be the area of the triangle." Unpacking this statement, we find the equation we've seen earlier for Heron's formula. To prove this result, Newton needed Euclid's proposition 8 from Book VI: Given a right-angled triangle, an altitude drawn to the hypotenuse will split the triangle into two similar right triangles. The similarity of the triangles allows us to build a proportion; for example, long leg is to short leg (x is to h) in one triangle as long leg is to short leg (h is to y) of the second triangle. Cross-multiplying, we get $h^2 = xy$. In other words, if we draw the perpendicular from the right angle to the hypotenuse, the square of that altitude's length is the product of the two parts of the hypotenuse, x and y, into which the altitude splits it.

Newton begins his proof with a triangle ABC. He then constructs a relatively simple figure with two similar triangles, lines extending in opposite directions to points that equal the distance from A to B, and a perpendicular of length h. From here, he applies the Pythagorean theorem to both triangles and sets the two resulting expressions equal: $c^2 - AD^2 = a^2 - DC^2$. The problem is now recognizable as a difference of squares, which can be factored into $(AD + DC)(AD - DC)$. Looking at Newton's construction, we see that $AD + DC$ is the whole base of the original triangle; in other words, it's b. Inserting that in the expression, we get $c^2 - a^2 = b(AD - DC)$, or $c^2 - a^2/b = AD - DC$. The proof continues with a series of steps that account for the various pieces of Newton's construction. In the end, we find that the expressions $2(s - a) = -a + b + c$, $2(s - b) = a - b + c$, and $2(s - c) = a + b - c$ appear in the formula that we've developed—at significant algebraic cost—for h^2. After a great deal of work, we arrive at h^2 (the area of the triangle squared) = $2(s - c) \times 2(s - b) \times 2s \times 2(s - a)/4b^2$. Simplifying, we get $h^2 = s(s - a)(s - b)(s - c)$; we take the square root to get the area of the triangle and bring to an end this conversation between Newton's algebra and Heron's geometry. ■

Suggested Reading

Gjertsen, *The Newton Handbook*.

Westfall, *Never at Rest*.

Whiteside, *The Mathematical Works of Isaac Newton*.

Questions to Consider

1. As noted in the lecture, Newton included the following problem in his *Arithmetica Universalis* of 1707: Find an expression for the hypotenuse (h) of a right triangle whose area is A and whose perimeter is P. Do the algebra to show that the answer is $h = \frac{P}{2} - \frac{2A}{P}$.

2. Did you find Newton's proof of Heron's formula to be brilliant or tedious? Could you see ahead of time where he was going with his multiple algebraic manipulations? Most people who can follow this step-by-step argument are nonetheless surprised when Heron's formula emerges, seemingly from nowhere, at proof's end. Nonetheless, it is an intriguing "conversation" between two great mathematicians from two different millennia.

The Legacy of Leibniz

Lecture 16

Remember Newton had his *anni mirabilis*, his miraculous years, when he's basically charting the course of modern science. Leibniz's miraculous years were in Paris, and they were miraculous indeed.

Gottfried Wilhelm Leibniz, born in Leipzig in 1646, was almost an exact contemporary of Newton and has been described as a universal genius. As a boy, the young Leibniz was obviously brilliant. He finished his undergraduate degree by about age 16 and had a doctorate in law by age 20. Unlike Newton, who spent at least the first part of his career as a professor at Cambridge, Leibniz spent all of his adult life as a public servant, but the extent of his interests and talents is breathtaking.

Leibniz, along with Descartes and Spinoza, is considered to be one of the three big names in 17th-century philosophy. He studied symbolic logic and worked on the binary system, which as we know, would become critical in the modern computer age. He was also a student of history, political theory, philology, physics, and engineering. In 1672, Leibniz was sent on a diplomatic mission to Paris, where he found a mentor, Christiaan Huygens, who could guide him in his mathematical training. Huygens was a Dutch mathematician and scientist who had invented the pendulum clock and discovered the rings of Saturn. Huygens challenged the young Leibniz to find the sum of the reciprocals of the triangular numbers: 1 + 1/3 + 1/6 + 1/10 + 1/15 … ? Leibniz discovered the finite answer to this infinite series, 2, and Huygens agreed to mentor him in mathematics.

Leibniz plunged into mathematical studies and, with his characteristic zeal, raced to the frontier of knowledge.

Leibniz plunged into mathematical studies and, with his characteristic zeal, raced to the frontier of knowledge. In 1674, he solved another infinite series that is now known as the Leibniz series: 1 – 1/3 + 1/5 – 1/7 + 1/9 – 1/11…. The value of this sum is $\pi/4$. In 1673 and 1676, Leibniz made diplomatic trips to London and was made a member of the Royal Society. During the later visit, he saw a manuscript of Newton's *De analysi*, and he later wrote to Newton, introducing himself and asking about Newton's work in mathematics. The two geniuses corresponded briefly.

In 1684, Leibniz consolidated his ideas on calculus and published the first paper on the subject, entitled "*A Nova Methodus*." For Leibniz, calculus was a set of rules for doing maximum and minimum problems, as well as tangents. In his 1684 paper, he stated the differential rules for sums, differences, products, and quotients. Two years later, he published the first paper on integral calculus.


The Teaching Company Collection.

The genius Leibniz earned his Ph.D. by the time he was twenty years old.

During the last years of his life, Leibniz became involved in a dispute with Newton over who created calculus, a subject we will cover in greater detail in the next lecture. Leibniz died in 1716, and unlike Newton's, his funeral was a small affair. He is, however, immortalized in a statue at the University of Leipzig. ■

Suggested Reading

Boyer, *The Concepts of the Calculus*.

Child, ed., *The Early Mathematical Manuscripts of Leibniz*.

Dunham, *The Calculus Gallery*.

Edwards, *The Historical Development of the Calculus*.

Grattan-Guinness, *From the Calculus to Set Theory*.

Hall, *Philosophers at War*.

Hoffman, *Leibniz in Paris*.

Newton, *The Correspondence of Isaac Newton*.

Questions to Consider

1. We saw how Leibniz cleverly summed the infinite series of reciprocals of the triangular numbers to get $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\cdots=2.$ But he didn't stop there. He also summed the reciprocals of the so-called "pyramidal numbers," which are those with denominators of the form $\frac{n(n+1)(n+2)}{6}$. Thus, the infinite series in question is $P=1+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}+\frac{1}{56}+\cdots$. Here's how Leibniz determined its value: First note that $\frac{2}{3}P=\frac{2}{3}+\frac{2}{12}+\frac{2}{30}+\frac{2}{60}+\frac{2}{105}+\frac{2}{168}+\cdots$. Now rewrite $\frac{2}{3}=1-\frac{1}{3}$, $\frac{2}{12}=\frac{1}{3}-\frac{1}{6}$, $\frac{2}{30}=\frac{1}{6}-\frac{1}{10}$, and so on and thereby show that $P=\frac{3}{2}$ *exactly*.

2. For those with a philosophical bent, review Leibniz's contributions to this field. You might also read up on how Leibniz's philosophy of optimism was satirized via the Dr. Pangloss character in Voltaire's *Candide*. There, Pangloss contends—in true optimist fashion—that ours is the "best of all possible worlds."

The Bernoullis and the Calculus Wars

Lecture 17

The Bernoullis were notoriously argumentative, combative, cantankerous, and contentious. They wanted always to have the glory aimed at them; they resented it when the glory was going elsewhere. I like to say that they were the kind of people that gave arrogance a bad name.

Neither Isaac Newton nor Gottfried Wilhelm Leibniz ever married or had children, but Leibniz had the next best thing in his two dedicated disciples, the Bernoulli brothers of Switzerland. Jakob Bernoulli, the older of the two, was a very accomplished mathematician. He investigated infinite series and published, posthumously in 1713, the *Ars Conjectandi*, *The Art of Conjecturing*, a treatise on probability theory. In particular, this work proved one of the foundational results of probability, the law of large numbers. With Leibniz and his younger brother Johann, Jakob also helped refine the subject of calculus.

Johann Bernoulli mentored a young lad named Leonhard Euler, the subject of our next lecture, and was hired to provide calculus lectures to the Marquis de l'Hospital, a French nobleman. L'Hospital later appropriated the written work Johann had sent him and published the first-ever calculus textbook in 1696, although he acknowledged that the work belonged to Leibniz and the Bernoulli brothers. Interestingly, the book covers similar topics as those seen in introductory calculus texts today. A max/min problem posed by l'Hospital is as follows: "Among all cones that can be inscribed in a sphere, determine that which has the greatest convex surface." Just as students do today, l'Hospital solved the problem by taking differentials and setting them equal to zero. The answer is that the height of the cone should be 2/3 the diameter of the sphere.

As you recall from the last lecture, the calculus wars developed between the British and Newton and the Europeans and Leibniz over who should get credit for this innovation in mathematics. In 1708, the Royal Society, headed by Newton, attributed the invention of "the Arithmetic of fluxions" to Newton himself, subtly levying a charge of plagiarism against Leibniz. Leibniz, of course, was infuriated and demanded an investigation to clear his name. Near the end of 1712, the Royal Society—again, headed by Newton—concluded that Leibniz's knowledge of calculus had come after his visit to London and his brief correspondence with Newton in 1676. Leibniz countered that Newton had known nothing of calculus until reading Leibniz's paper of 1684. The battle between the two giants raged on until the death of Leibniz in 1716. It's now clear that they both discovered calculus independently.

In 1696, Johann issued a challenge to solve the brachistochrone problem: … By 1697, when no one except Leibniz had solved the problem, Johann mailed it directly to Newton, who quickly found a solution.

The most famous battle in the calculus wars involved Johann Bernoulli, who came to be known as "Leibniz's Bulldog." In 1696, Johann issued a challenge to solve the brachistochrone problem: Assign to a mobile particle *M* the path *AMB*, along which, descending under its own weight, the particle passes from *A* to *B* in the briefest time. The answer, Johann noted, is not a straight path. By 1697, when no one except Leibniz had solved the problem, Johann mailed it directly to Newton, who quickly found a solution. In fact, the curve of quickest descent is the cycloid curve.

We might, perhaps, settle the calculus wars with a quotation from the mathematician Wolfgang Bolyai: "… it seems to be true that many things have, as it were, an epoch in which they are discovered in several places simultaneously, … just as violets appear on all sides in the springtime." So it was with calculus. ■

Suggested Reading

Boyer, *The Concepts of the Calculus*.

Dunham, *The Calculus Gallery*.

Edwards, *The Historical Development of the Calculus*.

Grattan-Guinness, *From the Calculus to Set Theory*.

Hald, *A History of Probability and Statistics and Their Applications before 1750*.

Hall, *Philosophers at War*.

Newton, *The Correspondence of Isaac Newton*.

Tent, *Leonhard Euler and the Bernoullis*.

Questions to Consider

1. Read about the wider Bernoulli clan. Besides the original brothers Jakob and Johann, other family members left their marks in mathematics and physics. Chief among these was Daniel Bernoulli (Johann's son), who gave us Bernoulli's principle in fluid dynamics, the principle behind the flight of airplanes and so much more.

2. An infinite series that especially intrigued the Bernoulli brothers was the so-called harmonic series: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$. Here's a clever argument by which Johann proved the series (in his words) "is infinite," which means its sum grows without bound: For the sake of contradiction, Johann assumed the series had a *finite* value, say *H*. He then grouped the series into pairs of terms, as follows: $H=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots=\left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}\right)+\cdots$ Within each pair, the first number is always larger than the second, i.e.: $1>\frac{1}{2}$, $\frac{1}{3}>\frac{1}{4}$, and so on. Thus, we get: $H=\left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}\right)+\cdots>\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{1}{6}\right)+\cdots$ from which it follows that $H > H$ (why?). This is obviously impossible for any finite quantity; thus, by contradiction, Johann concluded that the sum of the harmonic series cannot be a finite quantity.

3. There are many other proofs that the harmonic series diverges to infinity. You might want to find a few of these for the sake of variety.

Euler, the Master

Lecture 18

[His hardships make] Euler, really, the counterpart of Beethoven. Remember Beethoven loses his hearing and yet continues to produce great music. Euler loses his vision but continues to produce great mathematics.

Leonhard Euler, born in Basel, Switzerland, in 1707, was history's most prolific mathematician. His genius was evident from an early age. His father wanted him to become a pastor, but Euler's skills and talents seemed better suited to math and science. Thus, in 1720, when he was 13, it was arranged that he would study with Johann Bernoulli. In not too much time, Euler would pass his master, and Bernoulli would stand in awe of what his former student could do.

In 1722, at the age of 15, Euler graduated from the University of Basel; he then completed a master's degree. In 1727, at age 20, he took an appointment at the St. Petersburg Academy in Russia, then moved to the Berlin Academy in 1741. In 1766, he returned to St. Petersburg and remained there until his death in 1783.

Three of Euler's theorems were voted by mathematicians into the top five most beautiful theorems of all time, and almost 100 mathematical terms carry his name.

By all accounts, Euler had a phenomenal memory, which served him well after the 1730s, when he lost vision in one of his eyes, probably the result of an infection. In 1771, he lost vision in the other eye as a result of botched cataract surgery. Essentially blind, Euler nonetheless continued his active career. In 1775, for example, he produced 50 papers, dictating them to a group of young scribes. The collected works of his lifetime run to more than six dozen volumes—25,000 pages of published mathematics. The first volume of Euler's work was published in 1911, and the project to complete publication of his work is still ongoing a century later. The quality of his output is equally as astounding as the quantity. Three of Euler's theorems were voted by mathematicians into the top five most beautiful theorems of all time, and almost 100 mathematical terms carry his name. Further, Euler wrote great textbooks, including the *Introductio in analysin infinitorum* (1748), which introduced functions as the critical entity of mathematical study. He also wrote texts for differential and integral calculus, as well as books on mechanics, optics, and popular science. His bestselling book ever was *Letters to a German Princess*, a series of essays on elementary science.

Euler's blindness when he was in his sixties did not interfere with his prolific career.

Highlights of Euler's achievements include the number *e*, which denotes the base of the natural, or hyperbolic, logarithm. This number still plays a critical role in calculus. He also gave us Euler's identity, $e^{i\pi} + 1 = 0$, an expression that contains what are, perhaps, the five greatest numbers. The Euler polyhedral formula, V (vertices) + F (faces) = E (edges) + 2, is another famous result, pervasive in the realm of solids. Still another great result is the solution to Basel problem; here, the task is to find the exact sum of the following infinite series: 1 + 1/4 + 1/9 + 1/16 + 1/25 …. In the next lecture, we'll see Euler's solution, which is exactly $\pi^2 / 6$. What's called the Euler path was suggested by the question of whether the citizens of Königsberg could travel a route around their city in which they crossed each of its bridges once and only once. Euler's solution launched the discipline of graph theory. In geometry, he found something called the Euler line of a triangle, on which the orthocenter, the centroid, and the circumcenter align.

Condorcet, the great French mathematician, had this to say about Euler: "All celebrated mathematicians now alive are his disciples: there is no one who is not guided and sustained by the genius of Euler." ■

Suggested Reading

Biggs, et al., *Graph Theory: 1736–1936*.

Dunham, *Euler*.

Euler, *Elements of Algebra*.

———, *Introduction to Analysis of the Infinite*.

Fellman, *Leonhard Euler*.

Heyne and Heyne, *Leonhard Euler*.

Maor, *e: The Story of a Number*.

Richeson, *Euler's Gem*.

Sandifer, *How Euler Did It*.

Tent, *Leonhard Euler and the Bernoullis*.

Questions to Consider

1. Check the Euler polyhedral formula ($V + F = E + 2$) for polyhedra in the shape of the Great Pyramid of Cheops, the U.N. Building, and the Pentagon. Recall that V is the number of vertices (corners), F is the number of faces, and E is the number of edges in the polyhedron.

2. If you like geometrical constructions, get out a large piece of paper, draw a big triangle on it, and proceed to construct the orthocenter (where the three altitudes meet), the centroid (where the three medians meet), and the circumcenter (where the three perpendicular bisectors meet). If you do this carefully, you should see that these three points line up, with the centroid half as far from the circumcenter as from the orthocenter. This is the Euler line. Isn't it amazing that no one had spotted this before he did in 1767?

Euler's Extraordinary Sum

Lecture 19

[This] quotation I don't have a source for, but I think it's apt. Somebody said, "*Talent* is doing easily what others find difficult. *Genius* is doing easily what others find impossible." By that definition, Euler—solver of the Basel problem—was indeed a genius.

In this lecture, we look at Euler's solution to the Basel problem, issued in 1689 by Jakob Bernoulli. The challenge was to find the exact value of the infinite series 1 + 1/4 + 1/9 + 1/16 + 1/25…, the sum of the reciprocals of the squares. Bernoulli himself was able to get an approximation of the sum of this series: a value that is less than 2. In the 1730s, Euler found a more accurate approximation for the series: 1.644934…. He almost stopped there but later wrote, "… against all expectations I have found an elegant expression for the sum of the series. …"

To follow Euler's thinking, we need three preliminaries. First, we have to know: For which values of x is the sine of x equal to 0? Delving a little into trigonometry, we find $\sin x = 0$ when $x = \pm 180$ degrees or ± 540 degrees. Converting to radians, we get: The sine of x is 0 if $x = 0$ radians $\pm\ \pi$ radians $\pm\ 2\pi$ radians, 3π radians …. The second thing we need is Newton's infinite series for the sine of x: $x - x^3/3! + x^5/5! - x^7/7!$ …. The final piece of the puzzle is from algebra: Suppose $P(x)$ is an infinite-degree polynomial. Suppose further that $P(0) = 1$ and $P(x) = 0$ have infinitely many solutions: $x = a$, $x = b$, $x = c$, and so on. Euler was perfectly comfortable factoring $P(x)$ as follows: $(1 - x/a)(1 - x/b)(1 - x/c)$ …. This type of factorization works for second-, third-, and fourth-degree polynomials; Euler extended it to a polynomial of infinite degree.

With this solution, Euler became famous around the world.

To evaluate the series in the Basel problem, we begin by introducing an infinite-degree polynomial: $P(x) = 1 - x^2/3! + x^4/5! + x^6/7!$ …. This is similar to the sine series but not exactly the same. We then factor this polynomial using the preliminaries. We find the factorization works with $P(0) = 1$. What about the solutions to $P(x) = 0$? If we multiply the denominators and the numerators in the polynomial by x, we get Newton's series for the sine of x. By trying to solve $P(x) = 0$, we're now saying that the sine of x/x should be 0. Recall that the sine of $x = 0$ when $x = 0 \pm \pi \pm 2\pi \pm 3\pi$ …. Removing 0 from the possible solutions, we're left with $\pm \pi \pm 2\pi$ and so on. We now factor $P(x)$ by the third preliminary, which results in $(1 - x/2\pi)(1 - x/{-2\pi})(1 - x/3\pi)(1 - x/{-3\pi})$ …. Cleaning that up a bit, we see that $P(x)$ is now expressed on the left side of the equation as an infinite sum and on the right as an infinite product. We can link the expressions by multiplying these binomials together two at a time. We now have $P(x)$, after it has been factored and multiplied back together, written as $1 + x^2[-1/\pi^2 - 1/4\pi^2 - 1/9\pi^2 \ldots]$. We equate the coefficients of x^2 in this expression and set them equal, then eliminate the negative signs. Solving, we find that the sum of the infinite series $1 + 1/4 + 1/9 + 1/16 \ldots$ is $\pi^2/6$. With this solution, Euler became famous around the world. ■

Suggested Reading

Dunham, *The Calculus Gallery*.

———, *Euler*.

Edwards, *The Historical Development of the Calculus*.

Euler, *Introduction to Analysis of the Infinite*.

Grattan-Guinness, *From the Calculus to Set Theory*.

Sandifer, *How Euler Did It*.

Questions to Consider

1. As we've seen, Euler proved that $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\cdots=\frac{p^2}{6}$. But he did more. For instance, he found the value of the series of reciprocals of the *odd* squares, as follows: Let $S=1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\cdots$ be the series whose value we seek. By splitting the *original* series into odd and even components, Euler reasoned that: $\frac{p^2}{6}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\cdots=\left(1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\cdots\right)+\left(\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\frac{1}{64}+\cdots\right)$, so $\frac{p^2}{6}=S+\frac{1}{4}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots\right)=S+\frac{1}{4}\left(\frac{p^2}{6}\right)$ (why?). From this, deduce, as Euler did, that $1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\cdots=\frac{p^2}{8}$.

2. Euler also examined the alternating series of reciprocals of squares: $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\frac{1}{49}-\frac{1}{64}+\cdots$ and showed that this summed to $\frac{p^2}{12}$. Use question 1 to show how he got this answer.

Euler and the Partitioning of Numbers

Lecture 20

In 1750, … [Euler] writes a paper called *De Numeris Amicabilibus* on amicable numbers, and in this paper, he finds 58 more pairs. The supply had gone from 3 to 61. He multiplied the number of known amicable pairs by 20 in one paper.

Before we look at this lecture's great theorem, we briefly survey number theory. Among Euler's achievements in this area was his discovery of 58 pairs of amicable numbers, that is, whole-number pairs in which each is the sum of the proper whole-number divisors of the other (e.g., 220 and 284). Until Euler explored amicable numbers in 1750, only three such pairs were known. Euler also answered the challenge of finding four different whole numbers, the sum of any two of which is a perfect square: 18,530; 38,114; 45,986; and 65,570.

Euler also answered the challenge of finding four different whole numbers, the sum of any two of which is a perfect square: 18,530; 38,114; 45,986; and 65,570.

Our great theorem for this lecture relates to the partitioning of numbers. To start, let $D(n)$ be the number of ways of writing the whole number n as the sum of distinct whole numbers. For example, $D(5)$ would be 3: 5; 4, 1; and 3, 2. Now let $O(n)$ be the number of ways of writing n as the sum of odd numbers that are not necessarily distinct; $O(5)$ would be 3 again: 5; 3, 1, 1; and 1, 1, 1, 1, 1. The values $D(8)$ and $O(8)$ are both 6. In fact, Euler found that for all whole numbers, $D(n)$ and $O(n)$ are always the same.

Euler broke the proof of this theorem into three parts. First, he introduced $P(x)$, which is $(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)\ldots$, an infinite product of binomials. $P(x)$ is equal to 1 plus the sum as n goes from 1 to infinity of a certain number of x^n's. How many x^n's? Exactly $D(n)$, that is, exactly the number of ways of decomposing n into distinct pieces.

The second part of the proof involves an infinite geometric series: $1 + a + a^2 + a^3$…. The sum of this series is $1/1 - a$. In his proof, Euler introduced $Q(x)$, which is $(1/1 - x)(1/1 - x^3)(1/1 - x^5)(1/1 - x^7)$. Note that the powers here are odd. To eliminate the denominators, we replace the $1/1 - x$ by a series using the formula for the infinite geometric series; thus, $1/1 - x$ becomes $1 + x + x^2 + x^3$…, an infinite series. The next expression is $1/1 - x^3$. We replace this fraction with the infinite series $1 + x^3 + x^6 + x^9$…. At this point, we have infinitely many infinite series.

We now rewrite the first expression, $1 + x + x^2 + x^3$…, as $1 + x^1 + x^{1+1} + x^{1+1+1}$…. The second expression, $1 + x^3 + x^6 + x^9$…, becomes $1 + x^3 + x^{3+3} + x^{3+3+3}$ …. $Q(x)$ is equal to the product of all these expressions. We then multiply out these infinitely many infinite series. Again, we find that $Q(x)$ is 1 plus the sum as n goes from 1 to infinity of a certain number of x^n's. How many x^n's? Exactly $O(n)$, that is, exactly the number of ways of decomposing n into odd summands.

Notice that $P(x)$ is equal to 1 plus the sum of $D(n)x^n$, and $Q(x)$ is equal to 1 plus the sum of $O(n)x^n$. Recall that we're trying to prove that $D(n)$ is always equal to $O(n)$. That would be true if $P(x)$ and $Q(x)$ were the same, but they don't appear to be. Euler showed, however, that they are the same by changing the original $P(x)$, $(1 + x)(1 + x^2)(1 + x^3)$…, to a fraction. Then, canceling terms, $P(x)$ becomes $1/(1 - x)(1 - x^3)(1 - x^5)$…, the original $Q(x)$; thus, $D(n)$ must equal $O(n)$ for all n. ■

Suggested Reading

Dunham, *Euler*.

Euler, *Introduction to Analysis of the Infinite*.

Ore, *Number Theory and Its History.*

Sandifer, *How Euler Did It*.

Weil, *Number Theory*.

Questions to Consider

1. Find $D(10)$ and $O(10)$, then find $D(13)$ and $O(13)$, where, as in the lecture, $D(n)$ is the number of decompositions of n into the sum of *distinct* whole numbers and $O(n)$ is the number of decompositions of n into the sum of *odd* whole numbers. Needless to say, in both cases, your values should be the same (as Euler proved they must be).

2. As Euler was nearing the end of his career, he shared the mathematical spotlight with Joseph-Louis Lagrange (1736–1813), who gave us the so-called Lagrange four square theorem. This shows that any whole number can be written as the sum of four or fewer perfect squares. The theorem, proved by Lagrange with an assist from Euler, is one of the most intriguing results in all of number theory. Check it for $n = 13$, $n = 28$, and $n = 115$. That is, write each of these numbers as the sum of four or fewer perfect squares.

Gauss—The Prince of Mathematicians

Lecture 21

Gauss kind of removed himself from the story [in his proofs], but on his behalf, people would say, "Well, wait a minute now; the architects of the great cathedrals don't leave the scaffolding up. You remove the scaffolding and you see the art—the cathedral behind." Gauss would remove all the extraneous material and leave just the gem of the theorem.

Carl Friedrich Gauss was born to an impoverished family in Brunswick, Germany, in 1777. As we saw with Euler, Gauss's mathematical gifts were obvious from a young age. In one famous story from his childhood, Gauss correctly added the first 100 whole numbers in a matter of seconds. He saw that adding 1 + 100 = 101, 2 + 99 = 101, 3 + 98 = 101, and so on. He multiplied 100 by 101, then divided the product to get 5,050. Obviously, he was no ordinary student.

By the age of 15, Gauss's academic training was being funded by the duke of Brunswick. The young Gauss read the works of Newton, Euler, and others and became inspired by the great predecessors who had blazed the path for him. As an adolescent, he kept a notebook with his mathematical discoveries and conjectures; this is now a treasure-trove of great results, anticipating mathematics that would come forth in the 19th century. Gauss's first great discovery occurred in 1796, when he expanded on the idea that various regular polygons can be constructed geometrically. In 1799, he obtained his doctorate; his dissertation was the first proof of the fundamental theorem of algebra: Any real polynomial can be factored into the product of real linear and/or real quadratic factors. In 1801, he published his *Disquisitiones arithmeticae*, a deep and difficult book on number theory. Also in 1801, Gauss became famous for finding the "missing" asteroid Ceres using mathematics.

In 1807, Gauss became the director of the observatory at Göttingen in Germany and remained there for the rest of his career. He continued to make many contributions to mathematics and the sciences, including the theory of least squares, which underpins the subject of regression, and the

Gaussian, or normal, distribution in probability, which permeates all of statistical analysis.

As mentioned earlier, Gauss expanded the idea that regular polygons could be constructed with a compass and straightedge. In particular, he dealt with the heptadecagon, the regular 17-sided polygon. Earlier, Descartes had shown that it was possible to construct geometrically any length that is built from whole numbers and the operations of addition, subtraction, multiplication, division, and extraction of square roots. Expressions built in this way are now called quadratic surds (surds means "roots"). Gauss asserted that he could construct the 17-gon from the construction of $cos 2\pi/17$. Using something called the 16th-degree cyclotomic polynomial, Gauss concluded that $cos 2\pi/17$ could be written as a quadratic surd, which means that it is constructible. The generic statement of Gauss's theorem here is as follows: If $P = (2^2)^n + 1$ is prime, then a regular P-gon can be constructed with a compass and straightedge. Gauss's insights into such constructions and many other areas of mathematics opened doors for explorations that were fundamental to abstract algebra in the 19th century. ■

Gauss expanded the idea that regular polygons could be constructed with a compass and straightedge.

Suggested Reading

Dunnington, *Carl Friedrich Gauss*.

Ore, *Number Theory and Its History*.

Tent, *The Prince of Mathematics*.

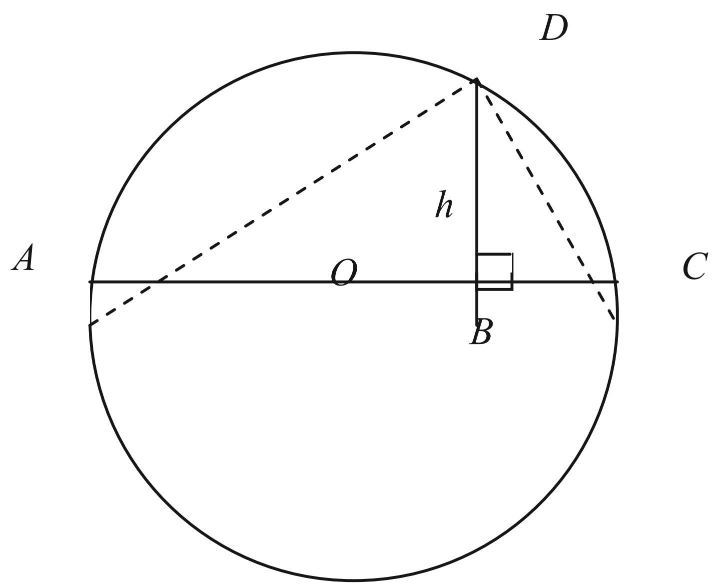
Trudeau, *The Non-Euclidean Revolution*.

Questions to Consider

1. Use the trick attributed to little Carl Friedrich Gauss to find the sum of the first *thousand* whole numbers. Without the trick, this is hideous; with it, it's a snap.

2. Here's how to construct the magnitude $\sqrt{5}$ with a compass and straightedge (indeed, any square root can be done in analogous fashion):

First, draw line AB of length 5 and extend it with segment BC of length 1. Thus, AC is of length 5 + 1 = 6 units. Bisect AC at O, and draw a semicircle with center O and with AC as the diameter, as shown below. From B, construct BD perpendicular to the diameter, with point D on the semicircle. All of these constructions can be accomplished with compass and straightedge.



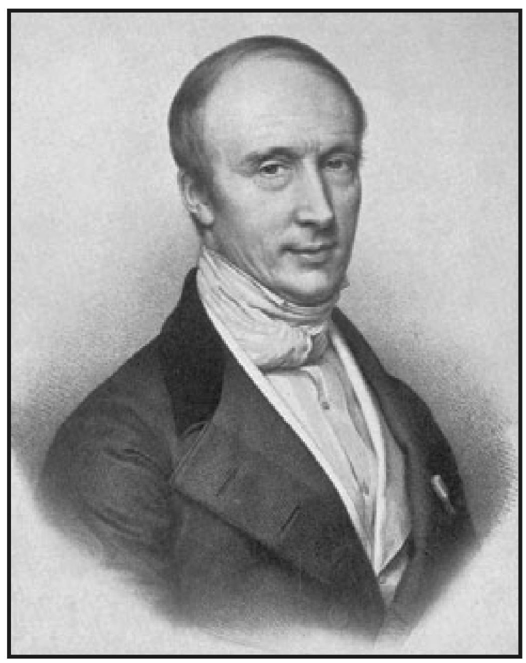
Now, $\angle ADC$ is a right angle (why?), and thus, ΔABD is similar to ΔDBC (see Lecture 4). Use the proportional sides of these similar triangles to show that h, the length of BD, is exactly $\sqrt{5}$, as required. Thus, if we can construct a magnitude with compass and straightedge, we can construct its square root as well.

The 19th Century—Rigor and Liberation

Lecture 22

What mathematicians decided was if you studied Euclid's geometry, where triangles all have 180 degrees, great. If you studied this non-Euclidean geometry, where they don't, great. … [B]ut certainly, they can't both be true. … Logically, [however,] these are sound, these are consistent, these are legitimate pursuits, and thus, mathematics is freed from the constraints of physical reality by the kinds of pursuits that Gauss, Bolyai, and Lobachevski began.

As we saw with the heroic 17th century, the 19th century was also a significant period in the history of mathematics. The achievements of this century can be broken into four themes: rigorization, liberation, diversification, and abstraction. We'll look at the first three of these in this lecture and the fourth in Lecture 23.



The Teaching Company Collection.

The French mathematician Cauchy defined the term "limit."

We begin with rigorization as it relates to the underlying ideas of calculus, particularly the notion of the derivative, which is the slope of the tangent line to a curve at a point. It shows how steeply the curve is rising or falling at that point. The slope of a line is the rise over the run, but we have only one point where the tangent occurs; thus, we usually approximate the slope by constructing a right triangle on the curve. But what if we want the exact slope of the tangent line? Newton used what he called "vanishing quantities"; that is, as the ratio of the change in y to the change in x ($\Delta y/\Delta x$) grows smaller and smaller, the slope is that ratio at the moment those values reach zero . Many found this definition a little hazy, as they did Leibniz's idea of infinitely small quantities. The process of rigorization to repair the foundations of calculus occurred in the 19th century, primarily through the efforts of the French mathematician Augustin-Louis Cauchy. Cauchy based the derivative on limits rather than vanishing or infinitely small quantities. For Cauchy, the derivative is the limit as Δx of the ratio $\Delta y/\Delta x$ goes to zero. Cauchy provided a definition of the term "limit" that was later refined by Karl Weierstrass.

One of the surprising results of non-Euclidean geometry is that different triangles have different angle sums; there is no single value for the sum of the angles of a triangle.

Also during this period, mathematics was freed from the bounds of reality. This liberation emerged in the study of non-Euclidean geometry. Recall that Gauss had explored triangles whose angles sum to less than 180 degrees. Two other mathematicians, the Hungarian Johann Bolyai and the Russian Nikolai Lobachevski, invented similar geometries at around the same time. One of the surprising results of non-Euclidean geometry is that different triangles have different angle sums; there is no single value for the sum of the angles of a triangle.

Finally, the third trend in the 19th century is diversification with regard to the mathematical players. Until this time, women were not supposed to pursue mathematics because the discipline was not seen as feminine. Further, formal education was denied to most women. One of the women who began to break down these barriers was Sophie Germain, a self-taught French mathematician. As a child, she had to hide her mathematical studies from her parents, and she was not allowed to attend university classes. Nonetheless, she carried out significant work in applied mathematics and number theory. Her most well known contribution is the Germain prime: A number p is a Germain prime if $2p + 1$ is also prime. Other women who followed Germain in the 19th century include Sophia Kovalevskaya of Russia and Grace Chisholm Young of Britain.

In our next lecture, we'll look at the fourth great trend of the 19th century, abstraction, as it appears in the set theory of Georg Cantor. ■

Suggested Reading

Boyer, *The Concepts of the Calculus*.

Child, ed., *The Early Mathematical Manuscripts of Leibniz*.

Dunham, *The Calculus Gallery*.

Edwards, *The Historical Development of the Calculus*.

Grabiner, *The Origins of Cauchy's Rigorous Calculus*.

Grattan-Guinness, *From the Calculus to Set Theory*.

Osen, *Women in Mathematics*.

Trudeau, *The Non-Euclidean Revolution*.

Those interested in the history of calculus might consult my article "Touring the Calculus Gallery," which can be found online at http://mathdl.maa.org/images/upload_library/22/Ford/dunham1.pdf. This describes in more detail the work of Cauchy, Weierstrass, and their colleagues who shored up the logical foundations of the calculus.

Cantor and the Infinite

Lecture 23

I find it impossible not to compare Cantor and Van Gogh. … They lived at roughly the same time. … They were both extremely innovative. … They were both undeniably geniuses, but they both faced criticism for their radical work and they both suffered their mental demons.

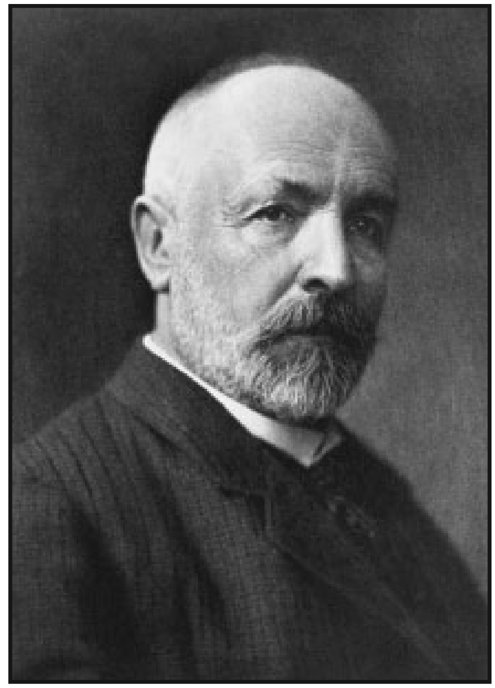
As we saw in the last lecture, the 19th century was a period of rigorization, liberation, and diversification in mathematics. It was also a time of abstraction, a trend that is most evident in the work of Georg Cantor and his abstract set theory.

Cantor's work rests on two foundational principles, the first of which is the "completed" infinite, as opposed to the "potential" infinite. The idea of the potential infinite is that no matter where you are, you can always go further. With the natural numbers, for example, we can start with 1, 2, 3 and go further to 4, 5, 6, and so on. Cantor said that we can legitimately discuss the completed infinite—we can treat the natural numbers as a complete set.

The idea of the completed infinite can present significant paradoxes.

The idea of the completed infinite can present significant paradoxes. Consider, for instance, the tennis ball game between Jeff and Mutt. We give Jeff two numbered tennis balls and tell him to toss away the one with the higher number. We give Mutt two numbered tennis balls and tell him to toss away the one with the lower number. As we continue the game, we see that after each round, Jeff and Mutt seem to be left with a number of tennis balls equal to the number of rounds that have been played. If we could complete this infinite game, Jeff would have infinitely many odd-numbered tennis balls. We might think that Mutt would also have infinitely many tennis balls, but instead, he's left with none because at each step of the game, he has thrown away the lowest-numbered ball. This is a somewhat uncomfortable conclusion.

According to Cantor, the completed infinite is a fixed constant quantity lying beyond all finite magnitudes. The other premise on which Cantor built his theory of the infinite was a definition of the equal cardinality of two sets: Two sets have the same cardinality (that is, contain the same number of items) if their members can be put into a one-to-one correspondence with each other. Cantor proposed to apply this definition to sets that aren't necessarily finite, such as the set of natural numbers (N) and the set of even numbers. It seems that the two sets wouldn't match up in a one-to-one correspondence, but we can pair every n in the natural numbers with $2n$ in the even numbers: 1 with 2, 2 with 4, 3 with 6, and so on. We can also find a one-to-one correspondence between the set of all integers (Z) and N. In Cantor's terminology, a set that can be put into a one-to-one correspondence with N is "denumerable."



Cantor explored the parodox of a completed infinite.

We can define the number 4 by saying that a set has four members if it can be put into a one-to-one correspondence with the faces on Mt. Rushmore. Under this definition, there were four Beatles. What if we want to define a number that is infinite? Cantor introduced the number aleph-naught ($\aleph_0$) as the number that can be put into a one-to-one correspondence with the natural numbers. It is a transfinite cardinal number—perfectly legitimate but beyond the finite numbers.

This new idea leaves us with two big questions: First, can every infinite set be put into a one-to-one correspondence with N? Second, is there any mathematical use for this exploration of infinity? We'll see Cantor's answers in the next lecture. ■

Suggested Reading

Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers*.

Dauben, *Georg Cantor*.

Grattan-Guinness, *From the Calculus to Set Theory*.

Russell, *The Autobiography of Bertrand Russell*.

Questions to Consider

1. Devise a modification of the tennis ball game with Jeff and Mutt and a third player—say, Emily—in which, after a completed infinitude of plays, Jeff is left with infinitely many tennis balls, Mutt with none, and Emily with exactly five. Does this sort of thing seem problematic, or are you comfortable (as was Cantor) with such outcomes?

2. Exhibit a formula that establishes a one-to-one correspondence between $\boldsymbol{N}$, the set of all natural numbers, and $\boldsymbol{O}$, the set of all odd natural numbers. Thus, your formula should match every whole number n with one and only one odd number and vice versa. In the terminology of Lecture 23, this shows that the set of odd whole numbers is denumerable.

Beyond the Infinite

Lecture 24

What Cantor has done here is found an infinite set that's bigger than *N*. *N* is the natural numbers—lots of infinite sets can be matched with it, but the set of all real numbers between 0 and 1 is so vast that it cannot be matched with *N*.

We ended the last lecture with two questions: Can every infinite set be matched in a one-to-one fashion with the natural numbers, and what good is any of this abstract set theory to other realms of mathematics?

To answer the first question, consider the fact that a real number can be written as an infinite decimal. For instance, 1/3 can be written as 0.33333…; π can be written as 3.141592…. We will regard the real numbers as the infinite decimals. A real number between 0 and 1 can be written as an infinite decimal, but its integer part will be 0, for example, 0.*ABCDEF*….

Let *I* (interval) be the set of all real numbers between 0 and 1. Cantor proved, through contradiction, that there cannot exist a one-to-one correspondence between *N* (the set of natural numbers) and *I*. If we assume the opposite, that such a correspondence exists, we might match up the natural numbers and the real numbers between 0 and 1 as follows: 1 with 0.123123123…, 2 with 0.9950000…, 3 with 0.602319888…, and so on. Let's now introduce a real number c of the form $0.c_1c_2c_3\ldots$ and assume that the first decimal digit of c, c_1, is different from the first decimal digit of the number matched with 1. We choose 2 for c_1 and write c as $0.2c_2c_3\ldots$. Next, c_2 must be different from the second decimal digit of the number matched with 2. Again, we choose any number other than 9 and continue the process. The number we generate belongs to *I*, but we have built it in such a way that it cannot be matched up with the numbers we have already matched to the natural numbers. We have reached a contradiction; thus, the set *I* of all real numbers between 0 and 1 is not denumerable. This is an infinite set that is larger than the infinite set *N*.

Cantor found an application for this work in the idea of an algebraic real number: A real number is algebraic if it is the solution to a polynomial with integer coefficients. By this definition, 2/3 qualifies as an algebraic number because it is the solution to the polynomial equation $3x - 2 = 0$. In fact, there are lots of algebraic numbers, some more complicated than others. Euler introduced the idea that there might be some real numbers that are not algebraic, that is, that could not be the solutions to any polynomial equation with integer coefficients. He called these non-algebraic numbers "transcendental" because they transcend algebra. Up until 1874, only two examples of transcendental numbers had been found; they seemed to be rare and difficult to establish. Then, Cantor wrote a paper in which he showed that the true situation is exactly the opposite: The transcendental numbers are by far more plentiful than the algebraic.

A landmark theorem has much in common with a landmark painting or a landmark novel. It possesses the aesthetic qualities of elegance and unexpectedness.

Cantor stands at the end of a long line of great thinkers that we have looked at in this course. In the first lecture, we noted that a landmark theorem has much in common with a landmark painting or a landmark novel. It possesses the aesthetic qualities of elegance and unexpectedness. I hope this course has indeed revealed to you the artistry of mathematics. ■

Suggested Reading

Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers*.

Dauben, *Georg Cantor*.

Dunham, *The Calculus Gallery*.

Grattan-Guinness, *From the Calculus to Set Theory*.

Russell, *The Autobiography of Bertrand Russell*.

Questions to Consider

1. Show that the number $1+\sqrt{5}$ is an algebraic number by exhibiting a specific polynomial equation with integer coefficients for which this number is a solution. This is done not by solving an equation but by "unsolving" one, as follows:

   Let $x=1+\sqrt{5}$. Then, $x-1=\sqrt{5}$ and, thus, $(x-1)^2=\left(\sqrt{5}\right)^2$. Expand and simplify this to get a quadratic equation with integer coefficients having $1+\sqrt{5}$ as a solution. This establishes that $1+\sqrt{5}$ is algebraic. NOTE: If you liked this exercise, apply similar reasoning to show that $\sqrt[5]{1+2\sqrt[3]{4}}$ is the solution to a (15^{th}-degree) polynomial equation with integer coefficients and, thus, is also an algebraic number. This number was mentioned in Lecture 24.

2. The set of all fractions—be they positive, negative, or zero—is called the set of *rational* numbers. Cantor proved that this is a denumerable set. Use this and a cardinality argument to explain why there must exist real numbers that are not rational. These, of course, are known as irrationals. In your proof, how many irrationals did you explicitly exhibit?

3. By course's end, did you have a favorite great thinker? A favorite great theorem?

Timeline

1850 B.C.	Moscow Papyrus.
fl. c. 600 B.C.	Thales.
c. 580–500 B.C.	Pythagoras.
c. 395/390–342/337 B.C.	Eudoxus, Greek mathematician who introduced the method of exhaustion.
fl. c. 300 B.C.	Euclid, author of *The Elements.*
287–212 B.C.	Archimedes, author of the *Measurement of a Circle* and other texts.
fl. c. 62	Heron of Alexandria.
c. 100–170 A.D.	Ptolemy, author of *The Almagest.*
c. 500	Development of the numerals in the base-10 system by Indian mathematicians.
529	Closing of the Library of Alexandria.
641	Much of the materials of the Library of Alexandria lost in a fire.
8th–13th centuries	Golden age of Islamic mathematics.
c. 780–850	Muhammad Mūsā ibn Al-Khwārizmī.
836–901	Thābit ibn Qurra, Islamic mathematician.

1048–1131	Omar Khayyam, Islamic mathematician and author of *The Rubiyat.*
1140	Translation of Al-Khwārizmī's work into Latin by Robert of Chester.
c. 1200	Introduction of Hindu-Arabic numerals to Europe.
1465–1526	Scipione del Ferro of Bologna, solver of the depressed cubic equation.
1482	*The Elements* printed on a mechanical press.
1494	Publication of the *Summa de Arithmetica* by Luca Pacioli.
1499/1500–1557	Niccolo Fontana of Brescia, nicknamed Tartaglia, "the Stammerer," Renaissance mathematician; challenged by Antonio Fiore to solve the depressed cubic equation.
1501–1576	Gerolamo Cardano.
1535	Tartaglia independently discovers the solution of the depressed cubic.
1539	Tartaglia gives the secret of the depressed cubic to Cardano.
1540–1603	François Viète, mathematician who introduced modern algebraic notation in his book *The Analytic Art* (1591).
1545	Publication of Cardano's *Ars Magna.*

1550–1617 .... John Napier, co-developer of logarithms.

1561–1630....................................... Henry Briggs, co-developer of logarithms.

1596–1650....................................... René Descartes.

17th century....................................... The heroic century of mathematics.

1623–1662....................................... Blaise Pascal.

1624... Publication of Briggs's logarithm tables in the *Arithmetica Logarithmica.*

1630–1690....................................... Pierre de Fermat.

1637... Publication of Descartes' *Discourse on Method.*

1642–1727....................................... Isaac Newton.

1646–1716....................................... Gottfried Wilhelm Leibniz.

1654–1705....................................... Jakob Bernoulli.

1665–1667....................................... Newton's *anni mirabilis*, "miraculous years."

1667–1748....................................... Johann Bernoulli.

1671... Newton made a member of the Royal Society based on his invention of the reflecting telescope.

1672... Leibniz sent on a diplomatic mission to Paris, launching his own "miraculous years" in mathematics under the tutelage of Christiaan Huygens.

1676.. Leibniz visits London and sees a manuscript of Newton's *De analysi.*

1684.. Leibniz publishes "*A Nova Methodus*," "A New Method," the first-ever published paper on calculus.

1687.. Publication of Newton's *Principia Mathematica.*

1689.. The Basel problem issued by Jakob Bernoulli: find the exact sum of the infinite series 1 + 1/4 + 1/9 + 1/16 + 1/25…; the problem was solved by Euler in the 1730s.

1696.. Publication of the first-ever textbook on calculus by the Marquis de l'Hospital, student of Johann Bernoulli; famous battle of the calculus wars initiated by Johann Bernoulli with his issuance of the brachistochrone problem.

1703.. Newton becomes president of the Royal Society.

1707.. Publication of Newton's textbook on algebra, the *Arithmetica Universalis*, which contained his proof of Heron's formula for the area of a triangle.

1707–1783...................................... Leonhard Euler.

1708.. The Royal Society, headed by Newton, attributes the development of calculus to Newton and obliquely accuses Leibniz of plagiarism.

1712 .... The Royal Society, still headed by Newton, restates its position attributing the development of calculus to Newton.

1713 .... Publication of the *Ars Conjectandi*, the first treatise on probability theory, by Jakob Bernoulli.

1736 .... Posthumous publication of Newton's treatise on "fluxions," i.e., calculus.

1740s .... Euler's work in the partitioning of whole numbers.

1748 .... Publication of Euler's *Introductio in analysin infinitorum*, a textbook on functions.

1750 .... Euler finds 58 pairs of amicable numbers.

1776–1831 .... Sophie Germain, Self-taught French mathematician.

1777–1855 .... Carl Friedrich Gauss.

1789–1857 .... Augustin-Louis Cauchy, French mathematician credited with the rigorization of calculus.

1793–1856 .... Nikolai Lobachevski, Russian mathematician who explored non-Euclidean geometry.

1796 .... Gauss's discovery of the regular 17-gon.

1799.. Gauss proves the fundamental theorem of algebra in his doctoral dissertation.

1801.. Publication of Gauss's *Disquisitiones arithmeticae*, a book on number theory.

1802–1860....................................... Johann Bolyai, Hungarian mathematician who explored non-Euclidean geometry.

1815–1897....................................... Karl Weierstrass, German mathematician who put forth the definition of limit.

1845–1918....................................... Georg Cantor.

1850–1891....................................... Sophia Kovalevskaya, Russian mathematician.

1868–1944....................................... Grace Chisholm Young, British mathematician and the first woman to receive a Ph.D. from a German university.

1874.. Publication of Cantor's remarkable theory of the infinite.

1896.. Discovery of Heron's work *Metrica* by R. Schöne in a library in Istanbul.

1911.. Start of the project to publish Euler's complete works by the Swiss Academy of Science; the project is still ongoing.

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