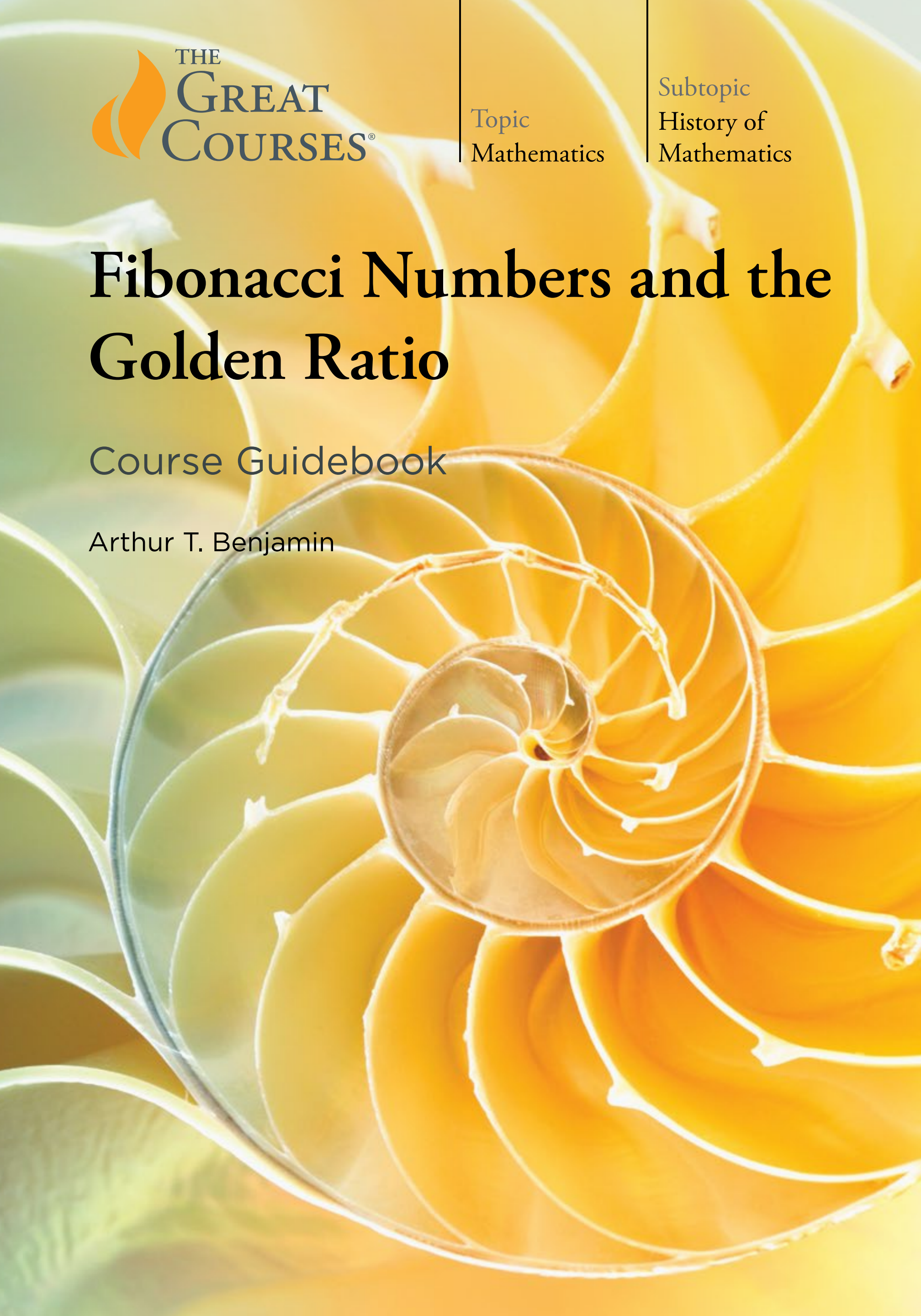

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Fibonacci Numbers and the Golden Ratio

Course Guidebook

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Acknowledgments

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1

Puzzling Fibonacci Patterns

Behold, the Fibonacci numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots.$$

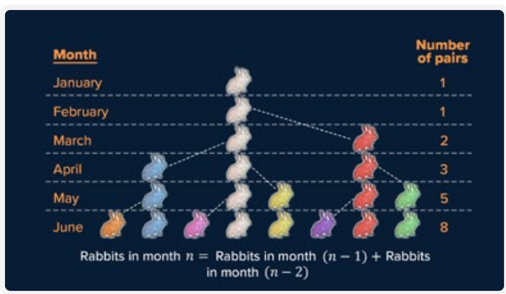
The Fibonacci numbers are as easy as $1 + 1 = 2$. This is followed by $1 + 2 = 3$, $2 + 3 = 5$, $3 + 5 = 8$, $5 + 8 = 13$, and so on. These numbers show up everywhere in the world of mathematics, as well as the world you live in, and have some truly magical properties.

The Fibonacci Numbers

The Fibonacci numbers were named after Leonardo of Pisa, later nicknamed Fibonacci, who was the greatest mathematician of the Middle Ages. Born around 1170, Fibonacci was the son of an Italian merchant who traveled with his father to the Middle East and Africa. It was there that he encountered the Indo-Arabic system of doing arithmetic and saw that it was vastly superior to the Roman numerals that were being used throughout Europe.

In 1202, Leonardo of Pisa wrote the book *Liber abaci* (*The Book of Calculations*), which taught the merchants of the Western world how to work with numbers the way they are used today. The book was filled with hundreds of mathematical exercises, and one of those problems had to do with immortal bunny rabbits.

The problem went something like this: Suppose that when a pair of bunny rabbits is born (one male and one female), after one month, the female becomes fertile and produces another pair of bunny rabbits the following month and every month thereafter, and the offspring have the same mating habits. If a new pair of bunny rabbits enters your garden in January, then how many pairs will there be in December?



In January, you have 1 pair—let's call them Adam and Eve—and you still have one pair, Adam and Eve, in February. But in March, that pair has a baby pair—let's call them Anna and Steve—so there are 2 pairs now. In April, Adam and Eve have another pair, so now there are 3 pairs. In May, Adam and Eve have another pair, but so do Anna and Steve, so now there are $3 + 2 = 5$ pairs. In June, all 5 pairs that were around in May are still around (since these rabbits never die), and the 3 pairs that were around in April have now had a pair of baby bunnies, so in June, there are $5 + 3 = 8$ pairs.

Continuing this process, in month n, there are all the pairs from month $n - 1$ plus all the babies of pairs that were around from month $n - 2$. Hence, the numbers will continue to grow in leapfrog fashion. July will have $8 + 5 = 13$ pairs of rabbits, August will have $13 + 8 = 21$, September will have $21 + 13 = 34$, October will have $34 + 21 = 55$, November will have $55 + 34 = 89$, and December will have $89 + 55 = 144$. These are the Fibonacci numbers!

January	1
February	1
March	2
April	3
May	5
June	8
July	13
August	21
September	34
October	55
November	89
December	144

As you'll discover in this course, the Fibonacci numbers are everywhere. Although inspired by an unrealistic problem involving immortal bunny rabbits, you'll learn that they have other real-world applications, and they show up in nature surprisingly often.

But perhaps the best thing about the Fibonacci numbers is the amazing patterns they display. This lecture explores many of these patterns, and subsequent lectures will explain these patterns as well as uncover more of them.

Summing Fibonacci Numbers

What do you get when you add the first 10 Fibonacci numbers: $1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55$?

If you add the first 3 Fibonacci numbers, you get $1 + 1 + 2 = 4$. The first 4 Fibonacci numbers—1, 1, 2, and 3—add up to 7. The first 5 of them—1, 1, 2, 3, and 5—add up to 12. If you add up 1, 1, 2, 3, 5, and 8, that gives you 20. Do you see a pattern?

$$1 + 1 + 2 = 4$$

$$1 + 1 + 2 + 3 = 7$$

$$1 + 1 + 2 + 3 + 5 = 12$$

$$1 + 1 + 2 + 3 + 5 + 8 = 20$$

The numbers 4, 7, 12, 20 are each 1 less than a Fibonacci number (5, 8, 13, and 21). In general, the sum of the first n Fibonacci numbers is 1 less than the $(n + 2)$nd Fibonacci number.

Just because the pattern works for the first few numbers, there's no guarantee that it'll work all the time. But the next lecture will offer several ways to prove that this pattern will persist forever.

Squaring Fibonacci Numbers

If you square the first few Fibonacci numbers, you get this:

$1^2 = 1$

$2^2 = 4$

$3^2 = 9$

$5^2 = 25$

When you add consecutive Fibonacci numbers, you always get the next Fibonacci number. (That's how they're created.) But you wouldn't expect something like that to work for their squares. But check this out:

$1 + 1 = 2$

$1 + 4 = 5$

$4 + 9 = 13$

$9 + 25 = 34$

And 2, 5, 13, and 34 are all Fibonacci numbers. The Fibonacci numbers are everywhere!

Here's another pattern. Suppose you add the first bunch of Fibonacci squares. What do you get?

$1 + 1 = 2$

$1 + 1 + 4 = 6$

$1 + 1 + 4 + 9 = 15$

$1 + 1 + 4 + 9 + 25 = 40$

$1 + 1 + 4 + 9 + 25 + 64 = 104$

Now, 2, 6, 15, 40, and 104 are not Fibonacci numbers, but the Fibonacci numbers are buried inside of them. Check it out: $2 = 1 \times 2$, $6 = 2 \times 3$, $15 = 3 \times 5$, $40 = 5 \times 8$, and $104 = 8 \times 13$.

$$1 + 1 = 2 = 1 \times \mathbf{2}$$

$$1 + 1 + 4 = 6 = 2 \times \mathbf{3}$$

$$1 + 1 + 4 + 9 = 15 = 3 \times \mathbf{5}$$

$$1 + 1 + 4 + 9 + 25 = 40 = 5 \times \mathbf{8}$$

$$1 + 1 + 4 + 9 + 25 + 64 = 104 = 8 \times \mathbf{13}$$

Here's another fun pattern. Take the square of any Fibonacci number—for example, 5^2, which is 25. Now take that Fibonacci number's neighbors and multiply them together. Here, the neighbors of 5 are 3 and 8, and their product, 3×8, is 24, which is almost 25—24 and 24 differ by 1.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...
1, 1, 4, 9, 25, 64, 169, 441, ...
$3 \times 8 = 24$

Let's do another example: 8^2 is 64, and 8's neighbors, 5 and 13, multiply to 65. Again they differ by 1.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...
1, 1, 4, 9, 25, 64, 169, 441, ...
$5 \times 13 = 65$

And this pattern apparently goes on forever.

But that's not all. Suppose you look at the neighbors that are 2 away. Here, the neighbors that are 2 away from 8 are 3 and 21, which multiply to 63, and that's still 1 away from 64.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

1, 1, 4, 9, 25, 64, 169, 441, ...

$3 \times 21 = 63$

Let's look at the number 13, whose square is 169. The neighbors that are 1 away have a product of 8 × 21, or 168, which is 1 below 169. And the neighbors that are 2 away from 13, which are 5 and 34, multiply to 170, which is 1 above 169.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

1, 1, 4, 9, 25, 64, 169, 441, ...

$8 \times 21 = 168$

$5 \times 34 = 170$

What if you look at the neighbors that are 3 away or farther? As you'll learn in a future lecture, there is still something very interesting that happens. Can you figure out the pattern now?

An Odd (or Maybe Even) Property

Which Fibonacci numbers are even?

The even numbers include 2, 8, 34, and 144. These occur in positions 3, 6, 9, and 12. In fact, every third Fibonacci number is even! And this property can be proven.

Notice that the Fibonacci sequence begins with numbers 1 and 1, which are both odd. So, the pattern begins odd followed by odd, so their sum must be even. Next, you have odd plus even, which is odd; then, even plus odd is odd. And now you're back to the starting point with two odd numbers, whose sum will be even. And this pattern—odd, odd, even, odd, odd, even—goes on forever. Thus, every third Fibonacci number is even, and the rest are odd.

What if you look at every 4th Fibonacci number? Starting with the number 3, you have 3, 21, 144, 987, and so on—all of which are divisible by 3. And indeed, every 4th Fibonacci number is a multiple of 3, and the rest are not multiples of 3.

The same sort of thing happens if you look at every 5th Fibonacci number: 5, 55, 610, and so on. Every 5th Fibonacci number is a multiple of 5, and in fact, they are the only multiples of 5.

1, 1, 2, 3, **5,** 8, 13, 21, 34, **55,** 89, 144,
233, 377, **610,** 987, 1597, 2584, 4181, ...

Likewise, every 6th Fibonacci number is a multiple of 8, and they are the only multiples of 8.

1, 1, 2, 3, 5, **8,** 13, 21, 34, 55, 89, **144,**
233, 377, 610, 987, 1597, **2584,** 4181, ...

And the pattern continues: Every 7th Fibonacci number is a multiple of 13, every 8th Fibonacci number is a multiple of 21, and so on.

1, 1, 2, 3, 5, 8, **13,** 21, 34, 55, 89, 144,
233, **377,** 610, 987, 1597, 2584, 4181, ...

1, 1, 2, 3, 5, 8, 13, **21,** 34, 55, 89, 144,
233, 377, 610, **987,** 1597, 2584, 4181, ...

This pattern says something interesting about prime Fibonacci numbers.

Prime Fibonacci Numbers

What can you say about the 66th Fibonacci number? Since every 6th Fibonacci number is a multiple of 8, then that means the 66th Fibonacci number has to be a multiple of 8. It also has to be a multiple of the 11th Fibonacci number, 89. But the point is that since it's divisible by 8, it can't be prime.

Applying this logic means that aside from the prime number 3 (which is the 4th Fibonacci number), every prime Fibonacci number must be in a prime position. For example, the prime Fibonacci numbers 5, 13, 89, 233, and 1597 occur in prime positions—namely, the 5th, 7th, 11th, 13th, and 17th positions.

It would be tempting to conjecture that all Fibonacci numbers in prime positions must be prime, but this pattern breaks down. For instance, even though the number 19 is prime, the 19th Fibonacci number, 4181, is not ($4181 = 37 \times 113$).

In fact, even though there are infinitely many prime numbers, only a few dozen prime Fibonacci numbers have been discovered. For example, even though there are 1229 prime numbers below 10,000, only 26 of the first 10,000 Fibonacci numbers are prime. The question remains: Are there infinitely many prime Fibonacci numbers, or is it a finite number?

The answer is that nobody knows! It's an unsolved problem in mathematics.

> It has been shown that among the Fibonacci numbers, there are only two perfect squares: 1 and 144 (12^2). And there are only two perfect cubes: 1 and 8 (2^3). And these proofs appeared only in the last few decades, relatively recently in the history of mathematics.

Dividing Consecutive Fibonacci Numbers

Let's look at what happens when you divide consecutive Fibonacci numbers. Specifically, let's divide each Fibonacci number by the Fibonacci number before it.

$$\frac{1}{1} = 1 \qquad \frac{2}{1} = 2 \qquad \frac{3}{2} = 1.5$$

$$\frac{5}{3} = 1.66\ldots \qquad \frac{8}{5} = 1.6 \qquad \frac{13}{8} = 1.625$$

$$\frac{21}{13} = 1.615384\ \ldots \qquad \frac{34}{21} = 1.619047\ldots$$

$$\frac{55}{34} = 1.617647\ldots \qquad \frac{89}{55} = 1.6181818\ldots$$

$$\frac{144}{89} = 1.617977\ldots \qquad \frac{233}{144} = 1.618055\ldots$$

Notice that these quotients are getting closer and closer to some number that is approximately 1.618. More specifically, as you go further and further in the Fibonacci sequence, the consecutive ratios are getting closer and closer to the number 1.6180339887—a remarkable number known as the golden ratio.

In fact, the entire second half of this course is devoted to the golden ratio, highlighting its mathematical properties and its appearance in geometry, art, nature, and the world around you. Indeed, just like the Fibonacci numbers, the golden ratio is everywhere.

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2

Proving Perplexing Properties

In the first lecture, you were introduced to the fascinating Fibonacci numbers and saw some of the astounding number patterns they possess. Although you saw lots of examples of these patterns, how can you be sure that the patterns will remain true forever and you're not just being fooled by a few examples? For instance, you learned that the first 10 Fibonacci numbers add up to 143, which is just 1 less than 144, the 12th Fibonacci number. To prove that this pattern does not break down, some notation and some simple-yet-powerful ideas are needed.

The Leapfrog Addition Property

Here are the first 12 Fibonacci numbers again:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots.$$

To better explore these numbers, let's denote the n^{th} Fibonacci number by F_n. In other words, the first Fibonacci number is F_1, the second Fibonacci number is F_2, and so on. Thus, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, and so on.

As you learned, the Fibonacci numbers have the leapfrog addition property; that is, starting with the third Fibonacci number, each number is the sum of the two previous numbers. In other words, for $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$. For example, if you say that $21 = 13 + 8$, that's the same as saying that $F_8 = F_7 + F_6$.

> Notice that $F_5 = 5$, which makes it easy to remember. A few other memorable Fibonacci numbers are $F_{10} = 55$ and $F_{12} = 144$.

One of the first patterns that was explored was that when you add the first 10 Fibonacci numbers—1, 1, 2, 3, 5, 8, 13, 21, 34, 55—their total is 143. In other words, when you sum the Fibonacci numbers F_1 through F_{10}, you get $F_{12} - 1$. And the claim is that this pattern always holds. In other words, for all values of n,

$$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1.$$

This says that the sum of the first n Fibonacci numbers is $F_{n+2} - 1$. This pattern can be proven in two different ways. The first way is with a technique called mathematical induction that will be applied frequently throughout this course.

Imagine that you have a bunch of dominoes on a table, one in front of the other, labeled 1, 2, 3, 4, and so on, and you want to convince someone that you can knock all of them down. To do so, you only have to prove two things: that you can knock over the first domino and that when a domino falls, so will the next one in front of it.

The idea behind proofs by induction is that you show that if a formula works for one value of n, then it must continue to work for the next value of n. Let's prove that if the first 10 dominoes fall, then so must the 11th; that is, suppose you know that the sum of the first 10 Fibonacci numbers, F_1 through F_{10}, add up to $F_{12} - 1$. Let's try to prove that this pattern will continue without doing any arithmetic.

If you know that the sum of the first 10 Fibonacci numbers is $F_{12} - 1$, then what will the sum of the first 11 Fibonacci numbers be? You want the answer to be $F_{13} - 1$, but how can you be sure?

Since you know that the Fibonacci numbers F_1 through F_{10} add to $F_{12} - 1$, then if you add F_{11} to both sides of that equation, you get that the Fibonacci numbers F_1 through F_{11} add to $F_{12} - 1 + F_{11}$.

$$F_1 + F_2 + F_3 + \dots + F_{10} = F_{12} - 1.$$

$$F_1 + F_2 + F_3 + \dots + F_{10} + F_{11} = F_{12} - 1 + F_{11}.$$

But from the leapfrog addition property, you know that $F_{12} + F_{11}$ is F_{13}, and this forces the right side of the equation to be $F_{13} - 1$.

$$F_{12} + F_{11} = F_{13}$$
$$F_1 + F_2 + F_3 + \ldots + F_{10} + F_{11} = F_{13} - 1.$$

In other words, if the pattern works for the first 10 numbers, then it must continue to work for the first 11 numbers. For a formal mathematical proof, you just replace 10 with n. You start by supposing that the Fibonacci numbers F_1 through F_n add to $F_{n+2} - 1$ and end with $F_{n+3} - 1$, as desired:

$$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$$
$$F_1 + F_2 + F_3 + \ldots + F_n + F_{n+1} = F_{n+2} - 1 + F_{n+1}$$
$$F_1 + F_2 + F_3 + \ldots + F_n + F_{n+1} = F_{n+3} - 1.$$

Here's another proof of the same formula. You're interested in the sum

$$F_1 + F_2 + F_3 + \ldots + F_n,$$

which can be written as

$$1 + 1 + 2 + 3 + 5 + 8 + \ldots + F_{n-1} + F_n.$$

Notice that each Fibonacci number can be expressed as the difference of the next two Fibonacci numbers. In other words, the first 1 is 2 − 1, the next 1 is 3 − 2, then 2 is 5 − 3, then 3 is 8 − 5, and so on. Thus,

$$1 + 1 + 2 + 3 + \ldots + F_{n-1} + F_n.$$

can be expressed as

$$(2 - 1) + (3 - 2) + (5 - 3) + (8 - 5) + \ldots + \\ (F_{n+1} - F_n) + (F_{n+2} - F_{n+1}).$$

But notice how the 2 from 2 − 1 cancels with the −2 from 3 − 2. Likewise, the 3s cancel, and the 5s cancel. In fact, everything cancels except for something from the first and last term. All that remains is the F_{n+2} term from the last expression and the −1 from the first expression. Thus, when the dust settles, you get a total of $F_{n+2} - 1$, as predicted.

This approach is called telescoping sums because, like a telescope, it all collapses neatly.

Odd-Positioned Fibonacci Numbers

You know what you get when you add the first *n* Fibonacci numbers together. What if you add every other Fibonacci number? For instance, when you add 1 + 2 + 5, you get 8. When you add 1, 2, 5, and 13, you get 21. When you add 1, 2, 5, 13, and 34, you get 55. The pattern is unmistakable:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots.$$

When you add every other Fibonacci number, starting at the beginning and stopping wherever you want, you always seem to get the next Fibonacci number. In terms of the *F* notation, you saw that $F_1 + F_3 + F_5 = 8$, which is F_6.

$$F_1 + F_3 + F_5 = 1 + 2 + 5 = 8 = F_6.$$

When you add F_1, F_3, F_5, and F_7, you get 21, which is F_8.

$$F_1 + F_3 + F_5 + F_7 = 1 + 2 + 5 + 13 = 21 = F_8.$$

And if you sum *n* of these terms, what do you get? If you sum 3 terms, you're summing up to F_5, and you get F_6. When you sum 4 of these terms, you're summing up to F_7, and you get F_8. So, with *n* terms, you're summing up to F_{2n-1}, and you get the next Fibonacci number, F_{2n}. In other words, for $n \geq 1$,

$$F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n}.$$

Let's prove this in two ways: first by induction and then by another method. As before, suppose you know that the formula works when *n* is 10. That is, the sum of the first 10 odd-positioned Fibonacci numbers is F_{20}.

$$F_1 + F_3 + F_5 + \ldots + F_{19} = F_{20}$$

What would happen if you added the next odd-positioned Fibonacci number, F_{21}? If you add that to both sides, you get

$$F_1 + F_3 + F_5 + \ldots + F_{19} + F_{21} = F_{20} + F_{21} = F_{22}.$$

By the leapfrog property, this results in F_{22}. So, the pattern continues. As before, the general proof is the same idea, but you would replace 10 with n.

Let's prove this identity again without induction. You want to show that for any number n,

$$F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n}.$$

You can prove this using the strategy of turning this into a problem you already know how to solve. Notice that

$$F_1 = 1, F_3 = F_1 + F_2, F_5 = F_3 + F_4, F_7 = F_5 + F_6,$$

and so on, all the way up to F_{2n-1}, which is $F_{2n-3} + F_{2n-2}$:

$$F_{2n-1} = F_{2n-3} + F_{2n-2}.$$

Hence, the left side of the equation becomes

$$1 + (F_1 + F_2) + (F_3 + F_4) + \ldots + (F_{2n-3} + F_{2n-2}).$$

But this is just 1 plus the sum of the first $2n - 2$ Fibonacci numbers, and you already know how to do that sum. Remember, the first 10 Fibonacci numbers add up to $F_{12} - 1$, and the first n Fibonacci numbers add up to $F_{n+2} - 1$. So, the first $2n - 2$ Fibonacci numbers would add up to what? Using the $F_{n+2} - 1$ formula but replacing n with $2n - 2$, you get a total of $F_{(2n-2)+2} - 1$, which is $F_{2n} - 1$.

$$(F_1 + F_2) + (F_3 + F_4) + \dots + (F_{2n-3} + F_{2n-2}) = F_{(2n-2)+2} - 1 = F_{2n} - 1$$

When you add 1 to both sides, the sum becomes $1 + (F_1 + F_2)$ all the way up to $(F_{2n-3} + F_{2n-2})$. That's $1 + (F_{2n} - 1)$. The 1s cancel, and you're left with F_{2n}, which is what you wanted to show.

$$1 + (F_1 + F_2) + (F_3 + F_4) + \dots + (F_{2n-3} + F_{2n-2}) = 1 + F_{2n} - 1 = F_{2n}$$

As an exercise, see if you can find and prove a formula for the first n evenly positioned Fibonacci numbers: $F_2 + F_4 + F_6 + \dots + F_{2n}$. Once you find your formula, see if you can prove it by induction and then without induction. After you've done that, see if you can find a formula for when you alternately add and subtract Fibonacci numbers: What's the pattern for sums like $1 - 1 + 2 - 3 + 5 - 8 + 13$?

Sums of Squares

Let's look at one of the patterns from the previous lecture that involves the sum of the squares of the first *n* Fibonacci numbers. You learned, for example, that

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 = 40 = 5 \times 8 = F_5F_6$$

and

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 = 104 = 8 \times 13 = F_6F_7.$$

And in general, if you take the sum of the squares of the first *n* Fibonacci numbers, you get F_nF_{n+1}. That is,

$$F_1^{\,2} + F_2^{\,2} + \ldots + F_n^{\,2} = F_nF_{n+1}.$$

There's a proof by induction for this pattern. Suppose that the formula is true for the sum of the squares of the first 10 Fibonacci numbers. That is, suppose that

$$F_1^{\,2} + F_2^{\,2} + \ldots + F_{10}^{\,2} = F_{10}F_{11}.$$

If you add the next term, $F_{11}^{\,2}$, to both sides, then this becomes

$$F_1^{\,2} + F_2^{\,2} + \ldots + F_{10}^{\,2} + F_{11}^{\,2} = F_{10}F_{11} + F_{11}^{\,2}.$$

You can factor out an $F_{11}(F_{10} + F_{11})$, and $F_{10} + F_{11}$ is F_{12}, so you're left with $F_{11}F_{12}$. So, the pattern continues.

Cassini's Identity

Recall from the previous lecture that when you square a Fibonacci number and then look at the product of its neighbors, they always differ by 1. For example, when you square 3, you get 9, and its neighbors have a product of 2 × 5, which is 10.

Using F notation, this says that

$$F_4^{\,2} - F_3F_5 = 9 - 10 = -1.$$

Likewise, when you center on the number 5, you see that

$$F_5^{\,2} - F_4F_6 = 5^2 - 3 \times 8 = 25 - 24 = 1.$$

And when you center on the number 8, you see that

$$F_6^{\,2} - F_5F_7 = 8^2 - 5 \times 13 = 64 - 65 = -1.$$

This is known as Cassini's identity, which says that for any number $n \geq 1$,

$$F_n^{\,2} - F_{n-1}F_{n+1} = (-1)^{n+1}.$$

As an exercise, prove Cassini's identity by induction.

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Applications of Fibonacci Numbers

Even though the Fibonacci numbers appeared as the answer to an exercise from an ancient book written by Leonardo of Pisa at the beginning of the 13th century, it's recently been determined that these numbers were actually put to use more than 1000 years earlier by poets from India. Their question was, How many different rhythm patterns of length n are possible where each beat in the pattern has length 1 or length 2? To put it mathematically, the question is, How many ways can you express a given number as a sum of 1s and 2s where the order of the 1s and 2s matters?

Different Ways to Express Numbers

There is only 1 way to express the number 1—namely, as 1. There are 2 ways to express the number 2: either as 2 or as 1 + 1. Let's abbreviate these as 2 and 1 1. There are 3 ways to get a total of 3: 2 1, 1 2, and 1 1 1.

Let's define f_n to denote the number of ways to express the number n in terms of 1s and 2s where order matters. So far, you've seen that $f_1 = 1$, $f_2 = 2$, and $f_3 = 3$. You might expect f_4 to be 4, but it turns out that f_4 is 5. That is, there are 5 ways to get a total of 4—namely, 2 2, 2 1 1, 1 2 1, 1 1 2, and 1 1 1 1. And it turns out that $f_5 = 8$.

Let's prove that $f_5 = 8$ in 2 ways. First, let's just list them. There are 3 ways that start with 2: 2 2 1, 2 1 2, and 2 1 1 1. And there are 5 ways that start with 1: 1 2 2, 1 2 1 1, 1 1 2 1, 1 1 1 2, 1 1 1 1 1. Hence, the number 5 can be expressed in 3 + 5 = 8 ways, and therefore, f_5 is 8.

Why does this make sense? If you start with a 2, then the remaining numbers have to add up to 3, and since $f_3 = 3$, there are 3 ways to do that. Similarly, if you start with 1, then the remaining numbers have to add up to 4, and since $f_4 = 5$, there are 5 ways to do that. Hence, the number 5 can be represented in 3 + 5 = 8 ways, and therefore $f_5 = 8$.

By the same logic, it's guaranteed that $f_6 = 13$. Why?

How many ways that add up to 6 start with a 2? The remaining numbers have to add up to 4, and there are $f_4 = 5$ ways to do that. How many start with 1?

To get a total of 6, the remaining numbers have to add up to 5, and there are $f_5 = 8$ ways to do that. Therefore, $f_6 = f_4 + f_5 = 5 + 8 = 13$. And in general, $f_n = f_{n-1} + f_{n-2}$, so the Fibonacci pattern is guaranteed to continue.

Now let's compare these f numbers with the classic Fibonacci numbers (F): f_1 is 1, which is F_2; f_2 is 2, which is F_3; f_3 is 3, which is F_4. And in general, for $n \geq 1$, $f_n = F_{n+1}$. And for convenience, let's define f_0 to be F_1, which is 1. That's easy to accept because there's 1 way to add up to 0 by using the empty sum consisting of no 1s or 2s. That empty sum adds up to 0.

To summarize, if you let f_n count the ways that you can create the number n using 1s and 2s, then f_n is the Fibonacci number F_{n+1}.

Tilings of Dominoes and Squares

There's a more visually appealing way to represent these sums. Let's replace the 1s and 2s with squares and domino tiles, where squares have length 1 and dominoes have length 2.

For instance, $f_4 = 5$ counts the 5 ways to get a total of 4 using 1s and 2s, with 2 2, 2 1 1, 1 2 1, 1 1 2, and 1 1 1 1. Let's represent these by the following tilings of length 4 that use squares and dominoes. So, for example, instead of 2 2, the tiling is domino followed by domino. Instead of 2 1 1, it's domino, square, square; 1 2 1 is square, domino, square; 1 1 2 is square, square, domino; and 1 1 1 1 is square, square, square, square. All of these have length 4.

In other words, the Fibonacci number f_4 counts the ways to tile a strip of length 4 with squares and dominoes. And more generally, the Fibonacci number f_n counts the ways to tile a strip of length n with squares and dominoes. This is more than just a nice application of Fibonacci numbers; these tilings actually help you understand almost all of the number patterns they possess and help you find new patterns.

Combinatorial Proofs

Let's use tilings to explain the number pattern that came from the immortal bunny rabbit problem, where

$$F_1 + F_2 + \dots + F_{10} = F_{12} - 1.$$

Or, in terms of f notation,

$$f_0 + f_1 + \dots + f_9 = f_{11} - 1.$$

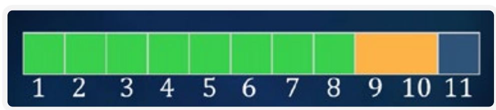
The strategy for proving this is to find a counting question that is answered by both sides of this equation, or identity. Let's propose the following question: How many tilings of length 11 contain at least 1 domino?

The answer is $f_{11} - 1$ because f_{11} counts all tilings of length 11 and you're just subtracting 1 to eliminate the tiling that contains all squares.

Let's try to answer this question another way. Since the tiling has at least 1 domino, let's focus on the location of the last domino. How many of these length-11 tilings end with a domino? So, there is a domino on cells 10 and 11. Such a tiling would look like this.

How many ways are there to tile cells 1 through 9? By definition, there are f_9 ways to do that.

Now let's move that domino back a cell. How many tilings where the last domino appears here?



To the right of the last domino is a square, and to the left of it are 8 cells to tile. How many ways are there to tile 8 cells? By definition, there are f_8 ways to do that.

If the last domino were here,

then everything to the right of it must be a square because it's the last domino, and the 7 cells to the left of it can be tiled in f_7 ways, and so on, all the way down to the case where the last domino is here,

where there's just 1 tiling ($f_0 = 1$) consisting of a single domino followed by all squares.

Altogether, depending on the location of the last domino, the number of tilings is $f_0 + f_1 + ... + f_8 + f_9$.

So, the question has now been answered in 2 ways. The question is, How many tilings of length 11 contain at least 1 domino?

Answer 1: $f_{11} - 1$.

Answer 2: $f_0 + f_1 + ... + f_8 + f_9$.

Since both answers are correct, they must be equal to each other. Thus, you've shown that

$$f_0 + f_1 + ... + f_8 + f_9 = f_{11} - 1.$$

Naturally, the same logic can be applied to prove the more general theorem that

$$f_0 + f_1 + ... + f_n = f_{n+2} - 1.$$

This kind of proof, where an equation or identity is explained by counting it in two different ways, is called a combinatorial proof.

> Just about every equation involving Fibonacci numbers can be explained using combinatorial proofs, as shown in the book *Proofs That Really Count: The Art of Combinatorial Proof* by Arthur T. Benjamin and Jennifer Quinn.

Zeckendorf's Theorem

The powers of 2 are as follows: 1, 2, 4, 8, 16, 32, 64, 128, ... This is another very useful sequence of numbers, and it's the basis for binary arithmetic.

Every positive integer can be uniquely expressed as the sum of distinct powers of 2. For example, $9 = 8 + 1$ and $100 = 64 + 32 + 4$. The way to express a number in terms of powers of 2 is called the greedy algorithm. Start with the largest power of 2 that you can use (in the example of 100, that would be 64) and see what's left over (in this case, $100 - 64 = 36$), and then you repeat the process: Take the largest power of 2 (that would be 32) and continue in this way.

A similar statement can be made about Fibonacci numbers. This is called Zeckendorf's theorem, which states that every positive integer can be uniquely represented as the sum of nonconsecutive Fibonacci numbers. And again, these numbers can be obtained through a greedy algorithm. For example, if you want to represent 100 in terms of nonconsecutive Fibonacci numbers, then the only way to do it is by taking 89, then 8, and then 3.

$$100 = 89 + 8 + 3$$

One fun application of Zeckendorf's theorem is converting kilometers into miles, and vice versa. For instance, to convert 100 kilometers into miles, you begin by writing 100 in terms of Fibonacci numbers. Zeckendorf's theorem says that $100 = 89 + 8 + 3$. Next, shift each Fibonacci number down to the next-lower Fibonacci number. The numbers 3, 8, and 89 become 2, 5, and 55. And when you add those numbers together, you get 62. So, 100 kilometers is approximately 62 miles. The true answer is about 62.1 miles.

Conversely, you can convert from miles to kilometers by shifting the Fibonacci numbers forward. For example, to convert 50 miles into kilometers, you write 50 as the sum of Fibonacci numbers, $34 + 13 + 3$, and then shift these Fibonacci numbers forward to get $55 + 21 + 5$, which is 81 kilometers.

The reason this works is that 1 mile is almost exactly 1.61 kilometers. So, to convert from miles to kilometers, you multiply by 1.61, and that's almost the same as the golden ratio, which is 1.618.... The method makes sense because the Fibonacci numbers essentially grow by this same factor.

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4

Fibonacci Numbers and Pascal's Triangle

Whereas the Fibonacci numbers are listed in one infinitely long row, the numbers in Pascal's triangle are listed in one infinitely long triangle. This triangle has a lot in common with the Fibonacci numbers. The Fibonacci numbers start with 1 and 1. In Pascal's triangle, the first and last entry in each row is 1. The Fibonacci numbers continue with $1 + 1 = 2$, and the first interior number in the triangle is 2, which comes from the sum of the two numbers above it, 1 and 1. In fact, the numbers in the interior of Pascal's triangle also behave like the Fibonacci numbers. As you know, each Fibonacci number bigger than 1 is the sum of the two numbers before it. Likewise, in Pascal's triangle, every number bigger than 1 is the sum of the two numbers above it. Like the Fibonacci numbers, the numbers in Pascal's triangle appear everywhere in mathematics.

The Rows and Columns of Pascal's Triangle

You may recall binomial expansions from algebra.

$$(x+1)^0 = \mathbf{1}$$

$$(x+1)^1 = \mathbf{1}x + \mathbf{1}$$

$$(x+1)^2 = \mathbf{1}x^2 + \mathbf{2}x + \mathbf{1}$$

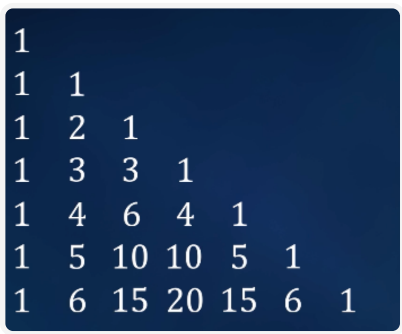
$$(x+1)^3 = \mathbf{1}x^3 + \mathbf{3}x^2 + \mathbf{3}x + \mathbf{1}$$

$$(x+1)^4 = \mathbf{1}x^4 + \mathbf{4}x^3 + \mathbf{6}x^2 + \mathbf{4}x + \mathbf{1}$$

If you look at the numbers that appear in these binomial expansions, what do you see?

It's the rows of Pascal's triangle! Because of this, the numbers inside of Pascal's triangle are called binomial coefficients.

Originally, Pascal's triangle was presented here as a centered isosceles triangle, but it can be easier to appreciate its patterns as a right triangle.



Just like in the original form, each row begins and ends with 1. The other numbers are the sum of the number directly above it and above it to the left.

Notice that the left-hand column has all 1s in it. That column is called column 0, and the top row is called row 0. The next column has all the positive integers. That column is called column 1, and the rows are labeled to match its integer.

The numbers in column 2—1, 3, 6, 10, 15, and so on—are called triangular numbers since with those many dots, you can create right triangles. For example, the 4th triangular number is

$1 + 2 + 3 + 4 = 10$, and the 5th triangular number is $1 + 2 + 3 + 4 + 5 = 15$.

Look at what happens when you add consecutive triangular numbers: $1 + 3 = 4$, $3 + 6 = 9$, $6 + 10 = 16$, $10 + 16 = 25$. Those numbers—4, 9, 16, and 25—are perfect squares: 2^2, 3^2, 4^2, 5^2. This makes sense since when you add two consecutive triangles, you get a perfect square. Notice that when you add the 4th triangular number to the 5th triangular number (that is, 10 and 15), you get 5^2, which is 25.

The number that appears in row n and column k is denoted as follows.

$$\binom{n}{k}$$

It's pronounced "n choose k" because it counts the ways that you can choose k objects from n different objects. These are also called combinations.

For example, the number that appears in row 5, column 2—which is 10—is equal to "5 choose 2" because you're counting the number of ways to pick 2 objects from a set of 5 different objects where order does not matter.

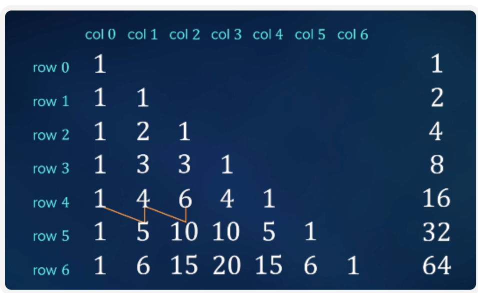
$$\binom{5}{2} = 10$$

From the set of numbers {1, 2, 3, 4, 5}, there are 10 ways to choose 2 different numbers: 1 2, 1 3, 1 4, 1 5, 2 3, 2 4, 2 5, 3 4, 3 5, and 4 5. (Notice that you don't include 2 1 because order doesn't matter, and 2 1 is the same as 1 2.) These 10 numbers are called the size-2 subsets of {1, 2, 3, 4, 5}. And they're counted by the number "5 choose 2."

Summing the Rows

Just like with Fibonacci numbers, the numbers in Pascal's triangle have all kinds of amazing patterns, and their counting interpretation can be used to explain them. For example, let's see what you get when you sum each of the rows of Pascal's triangle. The top row adds up to 1. The next row, 1 + 1, adds up to 2. The next row, 1 + 2 + 1, is 4. Then, they add up to 8, 16, 32, 64, …. These are powers of 2. Specifically, row n will add up to 2^n. For example, row 5 sums to 32, which is 2^5.

One way to understand this is to notice that each number in a given row contributes twice to the row below it. For example, the number 4 found in row 4 and column 1 contributes to $4 + 1 = 5$ and $4 + 6 = 10$.



As a result, the sum of each row should double each time. Consequently, if the numbers start out as powers of 2, then as they double, they will continue to be powers of 2.

Alternatively, you can take a counting approach. Let's look at row 5.

	col 0	col 1	col 2	col 3	col 4	col 5	col 6	
row 0	1							1
row 1	1	1						2
row 2	1	2	1					4
row 3	1	3	3	1				8
row 4	1	4	6	4	1			16
row 5	1	5	10	10	5	1		32
row 6	1	6	15	20	15	6	1	64

The numbers in row 5 count all possible subsets of the set {1, 2, 3, 4, 5}. A subset can be as small as the empty set with 0 elements, or it can have 1 element, or it could have 2 elements, and so on. In other words, there is 1 subset of size 0 (and that's denoted by "5 choose 0"). There are "5 choose 1"—5—subsets of size 1. There are "5 choose 2" subsets of size 2, "5 choose 3" subsets of size 3, "5 choose 4" subsets of size 4, and "5 choose 5"—1—subset of size 5.

$$\binom{5}{0} = 1 \quad \phi$$

$$\binom{5}{1} = 5 \quad 1, 2, 3, 4, 5$$

$$\binom{5}{2} = 10 \quad 12, 13, 14, 15, 23, 24, 25, 34, 35, 45$$

$$\binom{5}{3} = 10 \quad 123, 124, 125, 134, 135, 145, 234, 235, 245, 345$$

$$\binom{5}{4} = 5 \quad 1234, 1235, 1245, 1345, 2345$$

$$\binom{5}{5} = 1 \quad 12345$$

Altogether, the number of subsets is "5 choose 0" + "5 choose 1" all the way up to "5 choose 5."

$$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 2^5$$

But why should that equal 2^5? Another way to answer the question is to decide, for each element, whether to put the element in the subset or not. For instance, if you look at the elements 1, 2, 3, 4, 5 and decide yes, yes, no, no, yes, then that gives you the subset 125. And since you have 2 choices for the number 1 (either you include it in the subset or don't) and 2 choices for the number 2, 2 choices for the number 3, and so on, then the number of possible subsets is $2 \times 2 \times 2 \times 2 \times 2$, or 2^5.

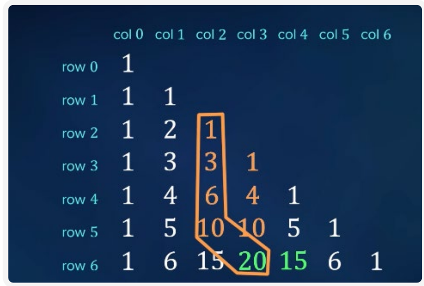
In general, if you're looking at the *n*th row of Pascal's triangle, you can do the same combinatorial proof. The question is, How many subsets are there of the set 1 through *n*? There are "*n* choose 0" subsets of size 0, "*n* choose 1" subsets of size 1, all the way up to "*n* choose *n*" subsets of size *n*. If you add these terms together, you get the sum of the *n*th row of Pascal's triangle.

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

Alternatively, you can decide for each element whether or not to include it in the subset, which can be done 2^n ways.

Summing the Columns

So, the rows add up to powers of 2. What can be said about the columns? Technically, each column has infinitely many numbers, so the sum would be infinity. But what if you add a bunch of numbers in the column, starting from the top, and then stop?



For instance, in column 2, if you add 1 + 3 + 6 + 10, you get 20, which is in the next row of the triangle. And that's not a coincidence—it happens everywhere in the triangle. As another example, 1 + 4 + 10 = 15. This is called the hockey stick identity because of the shape it makes.

Here's a combinatorial explanation of what's happening. The number 20 appears in row 6, column 3, so it's equal to "6 choose 3." The numbers being added in column 2 are "2 choose 2" + "3 choose 2" + "4 choose 2" + "5 choose 2." But why should that equal "6 choose 3"? You know that "6 choose 3" counts the size-3 subsets of the set 1 through 6. There are 20 of them.

1 2 3

1 2 4, 1 3 4, 2 3 4

1 2 5, 1 3 5, 1 4 5, 2 3 5, 2 4 5, 3 4 5

1 2 6, 1 3 6, 1 4 6, 1 5 6, 2 3 6, 2 4 6, 2 5 6, 3 4 6,
3 5 6, 4 5 6

These 20 subsets are arranged in 4 rows that contain 1 number, 3 numbers, 6 numbers, and 10 numbers, respectively. The last row has all the subsets that contain a 6. Notice that these subsets are all the size-2 subsets of {1, 2, 3, 4, 5} with a 6 tacked onto the end. Thus, there are "5 choose 2," or 10, such entries.

Similarly, the next row contains subsets that don't contain 6 but whose largest entry is 5. These are the size-2 subsets of {1, 2, 3, 4} with the number 5 tacked onto the end. Hence, there are "4 choose 2," which is 6, ways to do that. Likewise, the next row contains subsets whose largest entry is 4, and there are "3 choose 2," or 3, ways to do that, and the first row has the subset whose largest entry is 3, and there's just "2 choose 2," or 1, way to do that. As a result,

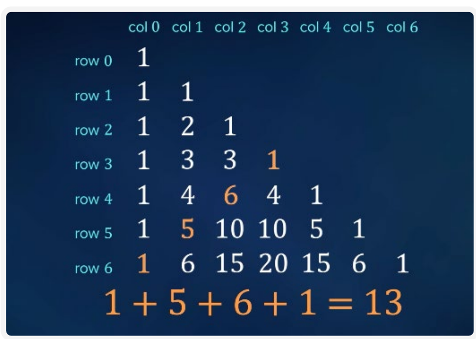
$$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}=\binom{6}{3}.$$

By essentially the same argument, you can prove the hockey stick identity for anywhere in the triangle. When you sum down column k, you get

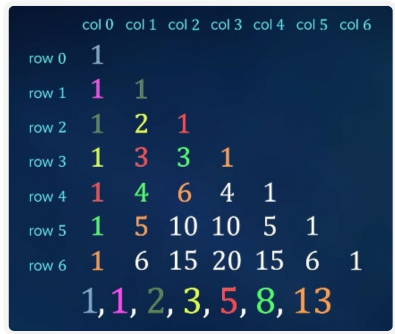
$$\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}.$$

Summing the Diagonals

Now that you've summed the rows and the columns of Pascal's triangle, let's add the diagonals, starting in the leftmost column and going up by 45 degrees. Starting in row 6, you get $1 + 5 + 6 + 1 = 13$.



Checking the earlier diagonals, you get totals of 1, 1, $1 + 1 = 2$, $1 + 2 = 3$, $1 + 3 + 1 = 5$, $1 + 4 + 3 = 8$, and $1 + 5 + 6 + 1 = 13$.



It's Fibonacci time! Where are these Fibonacci numbers coming from? Let's understand why $1 + 5 + 6 + 1 = 13$. In terms of the binomial coefficients, this is saying that

$$\binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} = f_6.$$

Why should that be? Remember, f_6 counts the ways to tile a strip of length 6 using squares and dominoes. Let's break these tilings into 4 cases, based on how many dominoes they have. Since the tiling has length 6, you can either have 0 dominoes, 1 domino, 2 dominoes, or at most 3 dominoes.

If there are 0 dominoes, then there's just 1 way to tile the strip using 6 squares (ssssss). Note that "6 choose 0" is 1, which is the first term in the sum.

0 dominoes: ssssss

$$\binom{6}{0} = 1$$

Next, suppose there's 1 domino. With 1 domino and a tiling of length 6, you'll need 4 squares to go along with that domino. That means you'll have 5 tiles: 1 domino and 4 squares. So, you take the 5 tiles and choose 1 of them to be a domino. Thus, you have 5 tilings.

1 domino: dssss, sdsss, ssdss, sssds, ssssd

$$\binom{5}{1} = 5$$

Again, note that the domino can either be the first, second, third, fourth, or fifth tile. Thus, there are "5 choose 1," or 5, tilings of length 6 that contain 1 domino.

If there are 2 dominoes, then you'll need 2 squares. They have length 6, which means there are 4 tiles being used. Notice that there are 6 such tilings.

2 dominoes: ddss, dsds, dssd, sdds, sdsd, ssdd

$$\binom{4}{2} = 6$$

Notice that the two dominoes can either be the first and second tile, or the first and third tile, and so on. You simply choose 2 of the 4 tiles to be dominoes, and that can be done in "4 choose 2," or 6, ways.

Finally, if a tiling of length 6 has 3 dominoes, then there are no squares. There's just 1 tiling: domino, domino, domino. And notice that "3 choose 3" = 1.

3 dominoes: ddd

$$\binom{3}{3} = 1$$

Thus, the number of length-6 tilings with 0, 1, 2, and 3 dominoes is, respectively,

$$\binom{6}{0}, \binom{5}{1}, \binom{4}{2}, \text{ and } \binom{3}{3}.$$

The same argument can be used that when you sum the *n*th diagonal, you must get the Fibonacci number f_n.

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \cdots = f_n$$

This pattern shows that the Fibonacci numbers are a fundamental part of Pascal's triangle.

Pascal's triangle can also be discovered inside of the Fibonacci numbers! You know that each Fibonacci number is the sum of the previous two. For instance, the leapfrog rule says that $f_{10} = f_9 + f_8$.

If you replace f_9 with $(f_8 + f_7)$ and f_8 with $(f_7 + f_6)$, this becomes

$$f_{10} = (f_8 + f_7) + (f_7 + f_6).$$

That's equal to $f_8 + 2f_7 + f_6$. Notice the coefficients are 1, 2, and 1.

If you apply the leapfrog rule to each of these terms—for instance, replace f_8 with $f_7 + f_6$ and f_7 with $f_6 + f_5$, and so on—you get

$$(f_7 + f_6) + 2(f_6 + f_5) + (f_5 + f_4).$$

And that's equal to $1f_7 + 3f_6 + 3f_5 + 1f_4$. And those coefficients are 1, 3, 3, 1.

Do you see the pattern? The coefficients are 1, 1; then 1, 2, 1; and then 1, 3, 3, 1. And these are rows 1, 2, and 3 of Pascal's triangle. More generally, for $n \geq 8$,

$$f_n = 1f_{n-4} + 4f_{n-5} + 6f_{n-6} + 4f_{n-7} + 1f_{n-8}.$$

There is an actual formula for "*n* choose *k*":

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where $n!$ is the product of all the numbers from 1 to n.

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5

A Favorite Fibonacci Fact

In 300 BCE, an important collection of mathematics books called Euclid's *Elements* were published. The Greek mathematician Euclid of Alexandria summarized and proved all the great mathematical ideas drawing on the works of Pythagoras and other great thinkers. It became the most successful textbook ever written and was used in classrooms well into the 19th century. The *Elements* consisted of 13 books, 9 of which covered geometry and 4 of which covered number theory. Many great theorems about the theory of numbers were presented and proved, and this was remarkable since ancient Greeks didn't have the convenient tools of algebra that are used today.

It's believed that, after the Bible, there have been more printings of Euclid's *Elements* than any other book ever written.

The Euclidean Algorithm

Let's start with some simple definitions and theorems. The integers consist of all the numbers, including 0, the positive numbers (1, 2, 3, 4, …), and the negative numbers (–1, –2, –3, …). The integers have lots of interesting properties, such as if you add or subtract or multiply two integers, you always get an integer result.

When you add or subtract two multiples of m, the answer is always a multiple of m. From a statement like $4 \times 3 = 12$, you can say things like 12 is a multiple of 4 and 12 is a multiple of 3. The numbers 3 and 4 are divisors of 12.

The addition theorem says that if m is a divisor of a and m is a divisor of b, then m is a divisor of $a + b$. Similarly, the subtraction theorem says that if m is a divisor of a and m is a divisor of b, then m is a divisor of $a - b$.

Here are all the (positive) divisors of 12: 1, 2, 3, 4, 6, and 12. Here are all the divisors of 17: 1 and 17. A number with only two divisors is called prime. So, 17 is prime since it's only divisible by 1 and itself.

The first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, and so on. (The number 1 is not technically prime since it has just one divisor—remember, a prime has to have two divisors. The number 1 is special because it divides every number, and it's called a unit.) The prime numbers are special because they can't be factored into smaller numbers. Numbers that can be factored into smaller numbers are composite. For example, 6 is composite since it's composed of 2 and 3: $6 = 2 \times 3$.

Euclid proved the fundamental theorem of arithmetic, which says that every positive integer can be written as the product of primes in a unique way. This is also called the unique factorization theorem.

For example, the number 12 can be factored into primes in just 1 way: $12 = 2 \times 2 \times 3$. (And that's the same as $3 \times 2 \times 2$ since order doesn't matter here.) The number 300 is $3 \times 10 \times 10 \times 10$, but each 10 can be factored as 2×5, so 3000 has prime factorization $2 \times 2 \times 2 \times 3 \times 5 \times 5 \times 5$.

In practice, prime factorizations are like a number's DNA, and if you know what it is, it tells you a lot about that number. For large numbers, it can be very hard to find. For instance, if someone multiplied two 1000-digit prime numbers and gave you the 2000-digit product, it is highly unlikely that you could ever find its prime divisors, even if you had a supercomputer working on it for a million years.

Again, here are all the divisors of 12: **1**, **2**, **3**, 4, **6**, and 12. And here are all the divisors of 30: 1, 2, 3, 5, 6, 10, 15, 30. The divisors they have in common are 1, 2, 3, and 6. The largest of these is 6, which is called the greatest common divisor (GCD) of 12 and 30.

$$\mathrm{GCD}(12, 30) = 6$$

When two numbers have a GCD of 1, it's said that they are relatively prime. Equivalently, two numbers are relatively prime if their prime factorizations have nothing in common.

You can find the GCD of two large numbers using a process that was described by Euclid thousands of years ago, appropriately named the Euclidean algorithm, which is believed to be the oldest numerical algorithm that's still in use today.

For example, to find the GCD of 847 and 203, you begin by dividing 203 into 847 and looking at the remainder. So, $847 = 4 \times 203 + 35$; in other words, 203 goes into 847 4 times, with a remainder of 35.

The claim is that any number that divides 847 and 203 must also divide 203 and 35. Let's see why. Suppose that d is a divisor of 847 and 203. You want to show that it's also a divisor of 203 and 35. Since d divides 203, it must also divide 4×203. But if d divides 847 and d divides 4×203, then by the subtraction theorem, it must also divide their difference. And $847 - (4 \times 203) = 35$. So, d divides 35.

By essentially the same argument (but by using the addition theorem instead), you can also show that anything that divides 203 and 35 must also divide 203 and 847. Thus, 847 and 203 have the same divisors as 203 and 35, which is a smaller pair of numbers. And if they have the same common divisors, they must have the same GCD. In other words, you've just shown that the GCD of 847 and 203 is the same as the GCD of 203 and 35.

$$\text{GCD}(847, 203) = \text{GCD}(203, 35)$$

Generally, to find the GCD of the numbers N and m, you divide N by m and replace N with the remainder. In other words, if $N = qm + r$, where q denotes the quotient and r denotes the remainder, then the Euclidean algorithm says that if $N = qm + r$, then the GCD of N and m is the same as the GCD of m and r.

$$\text{If } N = qm + r, \text{ then } \text{GCD}(N, m) = \text{GCD}(m, r).$$

So, you know that the GCD of 847 and 203 is the same as the GCD of 203 and 35. You might not know the factors of 203, but at least you now have a smaller problem. Next, you divide 35 into 203. When you do that, 35 goes into 203 5 times, with a remainder of 28, which means that the GCD of 203 and 35 is the GCD of 35 and the remainder, 28.

In summary, the Euclidean algorithm finds this GCD in just 4 steps. It starts with the GCD of 847 and 203, turns that into the GCD of 203 and 35, turns that into the GCD of 35 and 28, turns that into the GCD of 28 and 7, and turns that into the GCD of 7 and 0. And the GCD of 7 and 0 is 7 (because every number divides 0; 7 divides 0 since $7 \times 0 = 0$).

$$\text{GCD}(847, 203) = \text{GCD}(203, 35) = \text{GCD}(35, 28) =$$
$$\text{GCD}(28, 7) = \text{GCD}(7, 0) = 7$$

That's pretty fast! In fact, the Euclidean algorithm is one of the fastest algorithms around. It was also the first algorithm to be mathematically analyzed, in 1844 by Gabriel Lamé. It was the first result in the field of computational complexity theory, which is a major research area today.

Lamé showed that when the Euclidean algorithm computes the GCD of a and b, where a is bigger than b, then the number of steps is never more than 5 times the number of digits in b. This means that even if a and b are 100-digit numbers, even though you probably don't know the prime factors of a or b, you can find their GCD in at most 500 steps. That's in the blink of an eye for most computers.

The Fibonacci numbers result in the most steps. This is because $F_n = F_{n-1} + F_{n-2}$. In other words, F_{n-1} goes into F_n once with a remainder of $F_{n-2.}$ Thus, when you input the GCD of F_n and F_{n-1}, the output is guaranteed to be the GCD of F_{n-1} and F_{n-2}.

$$\mathrm{GCD}(F_n, F_{n-1}) = \mathrm{GCD}(F_{n-1}, F_{n-2})$$

By Euclid's algorithm, one pair of Fibonacci numbers always generates the next-lower pair, and the quotients are always 1 until you reach the final steps of the GCD of 2 and 1, which is the GCD of 1 and 0, which is 1. As a result, the numbers will decrease as slowly as possible.

A consequence of this is the following theorem: Consecutive Fibonacci numbers are relatively prime. That is, the GCD of consecutive Fibonacci numbers is always 1, so they'll never have any prime divisors in common.

The Messy Formula

Here are the first 12 Fibonacci numbers, along with their indices.

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}
1	1	2	3	5	8	13	21	34	55	89

Notice that F_3, F_6, F_9, and F_{12} are even. Another way to say this is that F_3 (which is 2) divides F_6, F_9, and F_{12}. You also learned the pattern that every fourth Fibonacci number is a multiple of 3. In other words, F_4 (which is 3) divides F_8 and F_{12}, and so on, and this pattern goes on forever.

Likewise, the claim is that F_5 will divide F_{10}, F_{15}, and so on. In general, the claim is that the Fibonacci number F_m is guaranteed to divide F_{2m}, F_{3m}, F_{4m}, and so on. To prove this, let's use the identity

$$f_{m+n} = f_m f_n + f_{m-1} f_{n-1}.$$

If you replace m with $m - 1$, you get

$$f_{m+n-1} = f_{m-1} f_n + f_{m-2} f_{n-1}.$$

And if you replace the fs with Fs (by shifting every index ahead by 1), you get

$$F_{m+n} = F_m F_{n+1} + F_{m-1} F_n.$$

Let's use this messy formula to prove that F_m divides F_{2m}. All you have to do is let $n = m$ in the messy formula and you get $F_{2m} = F_m F_{m+1} + F_{m-1} F_m$. And since both terms in the sum are multiples of F_m, then so is F_{2m}.

You've just shown that F_{2m} is a multiple of F_m. Similarly, you can prove that F_{3m} is a multiple of F_m, F_{4m} is a multiple of F_m, and so on. This is essentially a proof by induction.

To summarize, you've just proved that for any positive integer q, F_{qm} is a multiple of F_m. Another way of saying that is F_m always divides F_{qm}.

Feuclid's Algorithm

The GCD of 20 and 90 is 10. But what's the GCD of the 20th Fibonacci number and the 90th Fibonacci number? The 20th Fibonacci number is 6765. The 90th Fibonacci number is a 19-digit number: 2,880,067,194,370,816,120.

The GCD of F_{20} and F_{90} is 55, which is $F_{10,}$ the 10th Fibonacci number!

$$\text{GCD}(20, 90) = 10$$
$$\text{GCD}(F_{20}, F_{90}) = F_{10}$$

In general, the GCD of the mth Fibonacci number and the nth Fibonacci number is the gth Fibonacci number, where g is the GCD of m and n. More symbolically,

$$\text{GCD}(F_m, F_n) = F_g$$
$$g = \text{GCD}(m, n)$$

Alternatively,

$$\text{GCD}(F_m, F_n) = F_{\text{GCD}(m,n)}.$$

In other words, the GCD of the Fs is the F of the GCD!

Recall Euclid's algorithm:

$$\text{If } N = qm + r, \text{ then GCD}(N, m) = \text{GCD}(m, r).$$

Next, let's prove the following similar theorem:

If $N = qm + r$, then $\text{GCD}(F_N, F_m) = \text{GCD}(F_m, F_r)$.

Let's call this Feuclid's algorithm because it's exactly Euclid's algorithm, except you put Fs on top of everything!

How does Feuclid's algorithm prove the fact that the GCD of the Fs is the F of the GCD? When you apply Euclid's algorithm to N and m, you get that the GCD of N and m is the GCD of m and r:

$$\text{GCD}(N, m) = \text{GCD}(m, r).$$

And as you repeat Euclid, you eventually reach the GCD of g and 0, where g is the GCD of N and m, so it will look something like this:

$$\text{GCD}(N, m) = \text{GCD}(m, r) = \ldots = \text{GCD}(g, 0) = g.$$

When you apply Feuclid's algorithm, you get the same thing but with Fs on top:

$$\text{GCD}(F_N, F_m) = \text{GCD}(F_m, F_r) = \ldots =$$
$$\text{GCD}(F_g, F_0) = \text{GCD}(F_g, 0) = F_g.$$

The value for F_0, the 0th Fibonacci number, is 0 since that allows the pattern to continue: 0 + 1 = 1.

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6

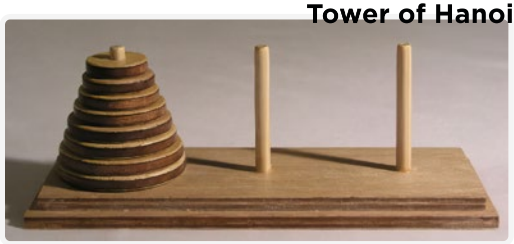
Fibonacci Numbers, Lucas Numbers, and Beyond

Where did the Fibonacci numbers get their name? They certainly were not named by the mathematician Leonardo of Pisa, who wrote about them in his book. He didn't even refer to himself as Fibonacci! That nickname (meaning "son of Bonacci") didn't really even exist until the 19th century. The person who is responsible for naming, popularizing, and proving many of the amazing properties of the Fibonacci numbers is the French mathematician Édouard Lucas.

The Lucas Numbers

Born in 1842, Édouard Lucas worked on many fun mathematical problems in number theory and recreational mathematics. For instance, he developed a very efficient way to determine if a large number is prime without looking for its factors. He wrote the first book on recreational mathematics, which introduced the famous Tower of Hanoi puzzle and was the first to publish a description of the dots-and-boxes game.

Tower of Hanoi



Lucas discovered many of the beautiful properties of the Fibonacci numbers and explored many variations of them as well. One sequence of numbers in particular, eventually called the Lucas sequence, could be thought of as the Fibonacci numbers' sibling since they have so much in common. Here are the Lucas numbers:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, \ldots.$$

Formally, $L_0 = 2$, $L_1 = 1$, and for $n \geq 2$, they satisfy the same Fibonacci leapfrog recurrence:

$$L_n = L_{n-1} + L_{n-2}.$$

The Lucas numbers have many patterns that resemble the Fibonacci number patterns.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
F_n	0	1	1	2	2	5	8	12	21	34	55	89	144
L_n	2	1	3	4	7	11	18	29	47	76	123	199	322

Recall that the first 10 Fibonacci numbers add to 143, which is $F_{12} - 1$, and the first n Fibonacci numbers add to $F_{n+2} - 1$. Similarly, the first 10 Lucas numbers add up to 321, which is the 12th Lucas number minus 1. That is,

$$L_0 + L_1 + L_2 + \dots + L_{10} = L_{12} - 1.$$

And in general, the first n Lucas numbers add up to the $(n + 2)$nd Lucas number minus 1. That is, similar to the Fibonacci numbers,

$$L_0 + L_1 + L_2 + \dots + L_n = L_{n+2} - 1.$$

You can prove this by induction or telescoping sums, just like you can with the Fibonacci numbers.

Every third Fibonacci number is even (F_0, F_3, F_6, F_9, and so on). The same is true with Lucas numbers; the even Lucas numbers are L_0, L_3, L_6, L_9, and so on.

Every fourth Fibonacci number is a multiple of 3 (F_0, F_4, F_8, F_{12}, and so on). With Lucas numbers, something similar happens, but it's not quite the same pattern. The first multiple of 3 is L_2 (which is 3), and this pattern continues with every 4 terms after that: L_6, L_{10}, L_{14}, and so on.

Every fifth Fibonacci number is a multiple of 5 (F_5, F_{10}, F_{15}, and so on). Yet with Lucas numbers, there are no multiples of 5—ever!

The square of a Fibonacci number is always 1 away from the product of its neighbors. For instance, $5^2 = 25$, and its neighbors, 3 and 8, multiply to 24. With Lucas numbers, the difference is always 5. For instance, $11^2 = 121$, and its neighbors, 7 and 18, multiply to 126. Notice that the neighbors 2 away also differ by 5. For instance, $4 \times 29 = 116$, and that's also 5 away from 121.

There's a pattern for the sum of the squares of the first few Fibonacci numbers. For example, $0^2 + 1^2 + 1^2 + 2^2 + 3^2 + 5^2 = 40$, which is 5×8. And more generally,

$$F_0^{\,2} + F_1^{\,2} + \ldots + F_n^{\,2} = F_n F_{n+1}.$$

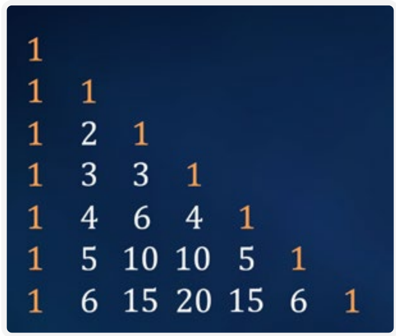
With Lucas numbers, there's something similar. For example, $2^2 + 1^2 + 3^2 + 4^2 + 7^2 + 11^2 = 200$, which is $11 \times 18 + 2$. And in general,

$L_0^{\,2} + L_1^{\,2}$ all the way up to $L_n^{\,2}$, we get $L_n L_{n+1} + 2$.

In lecture 4, you discovered the Fibonacci numbers hiding in Pascal's triangle when you summed the diagonals of Pascal's right triangle. When you sum the 6th diagonal, you find that $1 + 5 + 6 + 1 = 13$, the 6th Fibonacci number.

Are the Lucas numbers hiding in Pascal's triangle? Not exactly, but they show up when the triangle is modified in a natural way. The Fibonacci numbers start with 1 and 1, and each row of Pascal's triangle begins with 1 and ends with 1. The Lucas numbers start with 2 and 1. So let's begin and end each row with 1 and 2 and then fill out the middle of the triangle using the usual rule.

The diagonal starting in row 6 is $1 + 6 + 9 + 2 = 18$, which is the 6th Lucas number.

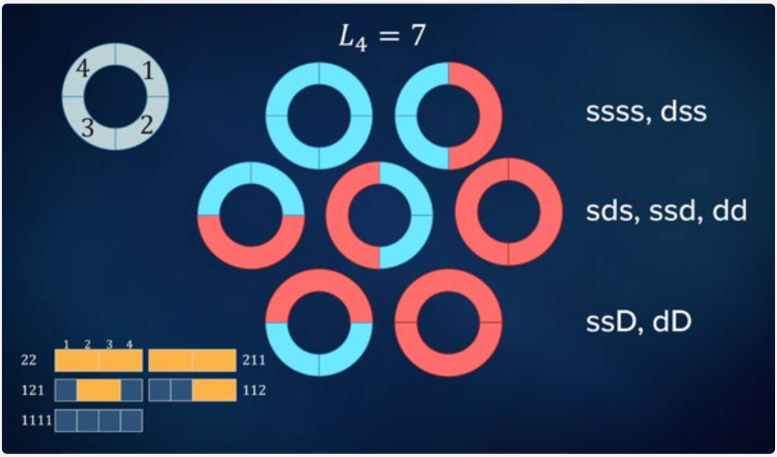


The other diagonals sum to 2, 1, 3, 4, 7, 11, 18, and so on. In general, the nth diagonal will add to the nth Lucas number.

The Lucas triangle has other interesting properties. For instance, just before each 2 are all the odd numbers: 1, 3, 5, 7, 9, 11. And before those are 1, 4, 9, 16, 25—the perfect squares.

Recall that the Fibonacci numbers had an excellent combinatorial interpretation. Naturally, since the Lucas numbers have so much in common with the Fibonacci numbers, then they must be counting something too. The Fibonacci number F_{n+1}, which can also be written as f_n, counts the ways to tile a strip of length n using squares and dominoes. For example, $f_4 = 5$ counts the tilings shown here.

So, what do the Lucas numbers count? It can be shown that the nth Lucas number, L_n, counts the ways to tile a circular strip of length n with squares and dominoes. These are called bracelets. For example, $L_4 = 7$ counts the following bracelets.



Notice that the bottom two bracelets have a domino that covers the clasp. And just as with Fibonacci numbers, this combinatorial interpretation allows the explanation of practically every pattern pertaining to Lucas numbers.

When Fibonacci and Lucas Numbers Interact

The fun begins when the Fibonacci numbers and the Lucas numbers interact with each other. For example, when you add two consecutive Fibonacci numbers, you get the next Fibonacci number. But what happens when you add two Fibonacci numbers that are 2 apart?

You see that $1 + 3 = 4$, $2 + 5 = 7$, $3 + 8 = 11$, and so on. The numbers 4, 7, and 11 are Lucas numbers! Using F notation, $3 + 8 = 11$ is the same as saying that $F_4 + F_6 = L_5$. In general,

$$F_{n-1} + F_{n+1} = L_n.$$

Or, using fs,

$$f_{n-2} + f_n = L_n.$$

What happens when you add two Lucas numbers that are 2 apart? Here's the data: $2 + 3 = 5$, $1 + 4 = 5$, $3 + 7 = 10$, $4 + 11 = 15$, $7 + 18 = 25$, $11 + 29 = 40$, and so on. It looks like you're always getting a multiple of 5. If you divide each term by 5, you get 1, 1, 2, 3, 5, 8, and so on. It's the Fibonacci numbers again! In general,

$$L_{n-1} + L_{n+1} = 5 \times F_n.$$

Here's another pattern. Look at what happens when you multiply F_n and L_n together. Looking at these vertical multiplications, you have $0 \times 2 = 0$, $1 \times 1 = 1$, $1 \times 3 = 3$, $2 \times 4 = 8$, $3 \times 7 = 21$. That's every other Fibonacci number! It turns out that $3 \times 7 = 21$ is the same as saying that $F_4 \times L_4 = F_8$, and $5 \times 11 = 55$ is the same as saying that $F_5 \times L_5 = F_{10}$. And in general,

$$F_n \times L_n = F_{2n}.$$

Recall that the $\text{GCD}(F_m, F_n) = F_{\text{GCD}(m,n)}$. But will this work for Lucas numbers? Is it true that the GCD of the *L*s is the *L* of the GCD? It turns out that it's sometimes true. In particular, it's always true when m and n are both odd. For example, you can find the GCD of L_{15} and L_{65} since 15 and 65 are both odd. The GCD of 15 and 65 is 5, and it follows that the $\text{GCD}(L_{15}, L_{65})$ will be L_5, which is 11.

More generally, the GCD of the *L*s will be the *L* of the GCD whenever m and n are divisible by the same power of 2. This happens when both numbers are odd, or when both numbers are divisible by 2 but not divisible by 4, or when both are divisible by 4 and not divisible by 8, and so on. Otherwise, the GCD will be 1 or 2.

Here's one more fun fact. If you look at the GCDs of F_n and L_n, they don't seem to have much in common. You know that every third Fibonacci and Lucas number is even, but it turns out that that's all they have in common. That is, if n is a multiple of 3, then the $\text{GCD}(F_n, L_n) = 2$. But if n is not a multiple of 3, then F_n and L_n are guaranteed to be relatively prime. That is, their GCD is 1.

A Few Fibonacci Variations

What is F_{-10}? In other words, to preserve the Fibonacci pattern, what number should precede 0?

You'd want $F_{-1} + F_0$ to equal F_1. That means $F_{-1} + 0$ should be 1, and therefore, F_{-1} should be 1. And when you add 1 to F_{-2}, you should get 0, so F_{-2} should equal −1. And when you add −1 to F_{-3}, you should get 1, so F_{-3} should be 2. Continuing this pattern, working backward, you have $F_{-4} = -3$, $F_{-5} = 5$, $F_{-6} = -8$, $F_{-7} = 13$, $F_{-8} = -21$, $F_{-9} = 34$, and $F_{-10} = -55$. You get the Fibonacci numbers again, but the signs are alternating. Notice that $F_{-10} = -55$ and $F_{10} = 55$, so their absolute values agree, and when n is even, F_{-n} is negative. When n is odd, F_{-n} is positive. So, in general,

$$F_{-n} = F_n(-1)^{n+1}.$$

Suppose that instead of adding the previous 2 numbers, you add the previous 3 numbers. These are called the Tribonacci numbers, and if you start with 0, 0, and 1, they continue as 1, 2, 4, 7, 13, 24, 44, and so on. The Tribonacci numbers also have many interesting properties, and they count the ways to tile a strip where you can use tiles of length 1, 2, or 3. And you can extend this further and look at the Tetranacci numbers, which are the sum of the previous 4 terms, and so on.

What other possible variations of the Fibonacci numbers exist? Let's go back just 2 terms, but now let's take twice the previous term plus 1 before that. Starting with 0 and 1, you get 2, then 5, then 12, then 29, then 70 (which comes from $2 \times 29 + 12$), and so on. These are called the Pell numbers, and they also have many amazing properties.

More generally, if you take two integers a and b and start with 0 and 1, then take a times the previous term plus b times the term before that, you get something called a Lucas sequence. When a and b are both 1, this results in the Fibonacci numbers. Practically every property of the Fibonacci numbers has a generalization with Lucas sequences.

The Lucas numbers 2, 1, 3, 4, 7, 11, and so on are not technically a Lucas sequence since they start with 2 and 1 instead of 0 and 1.

Suppose that $a = 4$ and $b = 9$. That is, let's start with $U_0 = 0$ and $U_1 = 1$ and then insist that $U_n = 4U_{n-1} + 9U_{n-2}$. That will give you a Lucas sequence of 0, 1, 4, 25, 136 (which comes from $4 \times 25 + 9 \times 4$), and so on.

Just as the Fibonacci number F_{n+1} counts tilings of length n with squares and dominoes, U_{n+1} does the same thing, but now the squares come in 4 different colors and the dominoes come in 9 different colors. Remarkably, as long as a and b are relatively prime, they will also obey the fact that the GCD of the Us will be the U of the GCD.

For most of the sequences, you've started with 0 and 1, but it's fine to start with any two numbers. For the Fibonacci numbers, you began with $F_0 = 0$, $F_1 = 1$, and after that, $F_n = F_{n-1} + F_{n-2}$. But suppose you start with two other numbers—for example, G_0 and G_1—and then apply the same leapfrogging recurrence that $G_n = G_{n-1} + G_{n-2}$. These are called the generalized Fibonacci numbers, or Gibonacci numbers. For example, if you let $G_0 = 10$ and $G_1 = 7$, then the Gibonacci sequence would go like this: $10 + 7 = 17$, $7 + 17 = 24$, $17 + 24 = 41$, and so on.

$$10, 7, 17, 24, 41, 65, 106, 171, 277, 448, \ldots$$

The Gibonacci numbers have all kinds of great properties too. You can analyze things like their sums, squares, and sums of squares.

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7

The Mysterious Golden Ratio

Using a calculator with a square root key, enter a number between 1 and 100. Add 1 to it and then take the square root. Your new number is probably between 1 and 10. Add 1 to it and take the square root. Do this 3 more times. If you've done your calculations correctly, then the first 2 digits of your answer should be 1.6. If you continue this process 5 more times, then the first 4 digits of your answer should be 1.618. And if you continue to do this calculation repeatedly, your number will get closer and closer to a number that starts 1.6180339887.... What is this number exactly?

The Solution to the Key Equation

In the long run, this is a number that's unchanged after adding 1 and then taking the square root. If you call that number x, when you take x, add 1, and then take the square root of it, you should get x. In other words,

$$x = \sqrt{x + 1}.$$

Squaring both sides, the number x has to satisfy

$$x^2 = x + 1.$$

This is called the key equation.

Notice that if you square 1.618, you get 2.617924, which is very close to 2.618. To get the number exactly, you solve $x^2 = x + 1$ by writing it as $x^2 - x - 1 = 0$ and then applying the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which tells you that x must equal

$$\frac{1 \pm \sqrt{5}}{2}.$$

In other words, x must either be

$$\frac{1 + \sqrt{5}}{2} = 1.618033 \ldots$$

or

$$\frac{1-\sqrt{5}}{2} = -0.618033 \ldots.$$

Since you know that the answer is positive, there's only one solution: 1.618033…, which is denoted by the Greek letter phi (ϕ). In other words,

$$\phi = \frac{1+\sqrt{5}}{2} = 1.618033 \ldots.$$

What makes ϕ so special? Since ϕ solves the key equation $x^2 = x + 1$, that means that

$$\phi^2 = \phi + 1 = 2.6180339887\ldots.$$

The reciprocal of ϕ, or $^1/_\phi$, is 0.6180339887…. Why? If you divide the equation $\phi^2 = \phi + 1$ by ϕ, you get that $\phi = 1 + {}^1/_\phi$, and therefore

$$\frac{1}{\phi} = \phi - 1 = 0.6180339887 \ldots.$$

In summary, if you add 1 to ϕ, you get ϕ^2. If you subtract 1 from it, you get $^1/_\phi$. And if you divide the equation $^1/_\phi = \phi - 1$ by ϕ again, you get that

$$\frac{1}{\phi^2} = 1 - \frac{1}{\phi} = 0.381900112 \ldots.$$

In other words, $1/\phi + 1/\phi^2$ is exactly equal to 1:

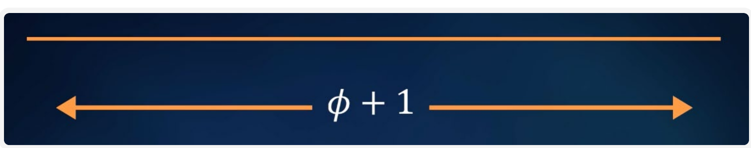
$$\frac{1}{\phi} + \frac{1}{\phi^2} = 1.$$

The Divine Proportion

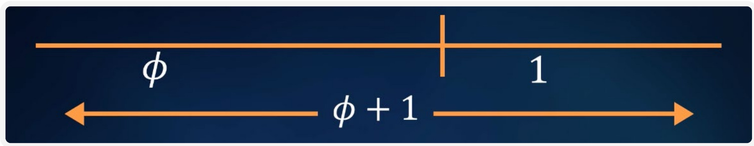
Around the year 1500, the Italian mathematician Luca Pacioli wrote a book dedicated to the golden ratio called *Divina proportione*, which translates to the *Divine Proportion*. Interestingly, Pacioli was also considered the father of modern accounting and bookkeeping, promoting the system of double-entry bookkeeping to the Europeans. He wrote a treatise called *De viribus quantitatus,* which translates to *On the Powers of Numbers*, which contained many number puzzles and was inspired by the book *Liber abaci*, the same book that introduced the Fibonacci numbers.

The illustrator of *Divina proportione* was Pacioli's friend Leonardo da Vinci, who had taken mathematics lessons from him. They even shared a house together in Florence, Italy. In this book, the author gives several reasons why ϕ should be considered "divine," including the number's simplicity; its irrationality, which represents God's incomprehensibility; the fact that it appears in a holy trinity; and how its self-similarity and inevitability invokes God's omnipresence and invariability.

Let's consider the holy trinity property. Imagine that you had a stick of length $\phi + 1$.



Let's split the stick into 2 pieces of length ϕ and length 1.



But notice that since $\phi + 1 = \phi^2$, the length of the stick is also ϕ^2.

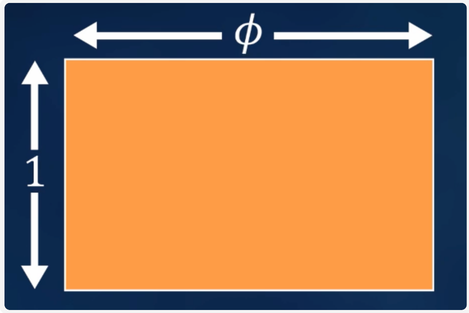
Thus, the ratio of the long piece to the short piece is $\phi/1$, which is ϕ. And the ratio of the total length to the long piece is ϕ^2/ϕ, which is also ϕ. In other words, this stick has the property that long/short = total/long. Isn't that cool? Some would say "divine"!

These lengths of 1, ϕ, and ϕ^2 are the holy trinity referred to by Pacioli and one of the reasons that he calls ϕ the divine proportion. This number is also known as the golden section.

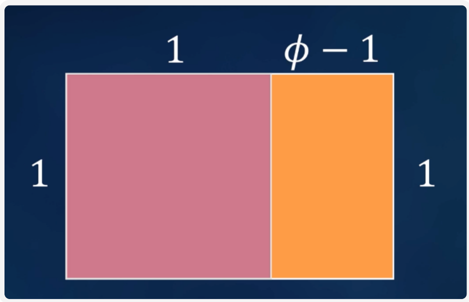
The same thing happens for sticks of any length, as long as the ratio of the long piece to the short piece is equal to ϕ. That's why ϕ is called the golden ratio. In fact, it can be shown that ϕ is the only ratio that satisfies this holy trinity property.

The Golden Rectangle

Let's transition from the 1-dimensional golden section to the 2-dimensional golden rectangle. A golden rectangle is any rectangle where the ratio of the long length to the short length is ϕ. For example, this rectangle has height 1 and width ϕ, which is about 1.618.

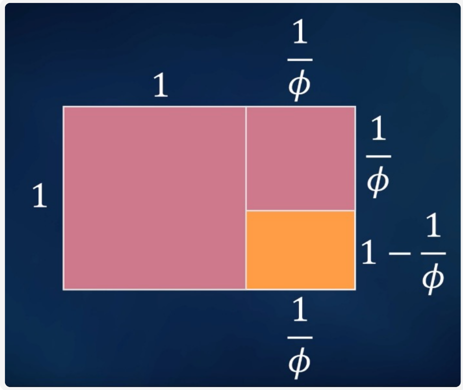


If you take this rectangle and remove a 1-by-1 square, what remains?



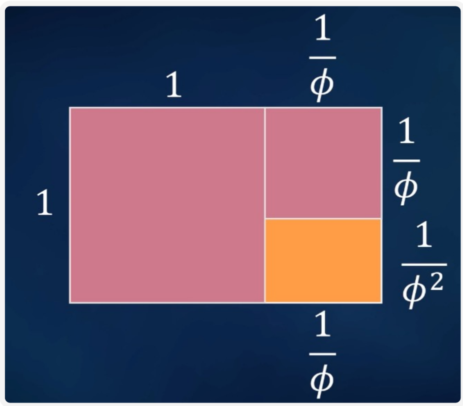
You're left with a rectangle with dimensions 1 (height) and ϕ – 1 (width). But recall that $\phi - 1 = {}^{1}/_{\phi}$, so the long edge of the remaining rectangle is still ϕ times longer than the shorter edge, just like in the original rectangle.

If you want, you can continue this process. You would remove a $1/\phi$-by-$1/\phi$ square, and the remaining rectangle has width $1/\phi$ and height $1 - 1/\phi$.



Because $1 - 1/\phi$ is equal to $1/\phi^2$, the new rectangle has long length $1/\phi$ and short length $1/\phi^2$, and once again, the long length is ϕ times longer than the short length.

And this process continues forever.



This is what Luca Pacioli referred to as the self-similarity property. When you start with a golden rectangle (where the long side is ϕ times longer than the short side) and remove the largest square inside it, the resulting rectangle will still be a golden rectangle!

Many people consider the golden rectangle to be the most beautiful rectangle. Many artists, architects, and advertisers deliberately incorporate this rectangle in their work.

Infinite Sums

Here's a formula that you may have seen in high school or college. It's called the geometric series. It says that for any number *x* between –1 and 1,

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}.$$

For example, if $x = ½$, then

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$$

If $x = 0.6$, then

$$1 + 0.6 + 0.6^2 + 0.6^3 + \cdots = \frac{1}{1-0.6} = \frac{1}{0.4} = 2.5.$$

Let's use a number that's a little more golden. Let's let g (as in *golden*) equal $^1/_\phi$, which is 0.618…. This number is a little bigger than 0.6, so you should get a sum that's a little bigger than 2.5. Here's what you start with:

$$1 + g + g^2 + g^3 + \cdots = \frac{1}{1-g}.$$

Remember, $g = {^1/_\phi}$, so that's

$$\frac{1}{1-\frac{1}{\phi}}.$$

And since you know that

$$1 - \frac{1}{\phi} = \frac{1}{\phi^2},$$

then this infinite sum is equal to

$$\frac{1}{\frac{1}{\phi^2}} = \phi^2.$$

This makes sense because $\phi^2 = 2.618...$, and you were expecting the answer to be a little higher than 2.5. Notice that since $g = {}^1/_\phi$, that means that $\phi = {}^1/_g$, so the sum ϕ^2 can also be written as ${}^1/_{g^2}$.

When $g = {}^1/_\phi$, you have the geometric series

$$1 + g + g^2 + g^3 + \cdots = \phi^2 = \frac{1}{g^2}.$$

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8

Phi: The Most Irrational Number

Just like the Fibonacci numbers, the golden ratio is everywhere! In this lecture, you'll discover how the golden ratio arises from the Fibonacci numbers. You'll explore messy-looking expressions called continued fractions, and you'll encounter proofs by contradiction and impossibility proofs. You'll prove that the ratio of Fibonacci numbers is guaranteed to get closer and closer to the number ϕ. You'll also prove that ϕ is irrational and discover that the golden ratio ϕ is the most irrational number.

Continued Fractions

Let's evaluate this continued fraction.

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}}$$

You can evaluate it by starting at the bottom and carefully work your way up to the top. First, at the very bottom,

$$1+\frac{1}{1}=2.$$

Then, the problem reduces to:

$$1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}.$$

Working from the bottom,

$$1+\frac{1}{2}=\frac{3}{2},$$

so that results in this:

$$1+\cfrac{1}{1+\frac{1}{\frac{3}{2}}}.$$

But

$$\frac{1}{\frac{3}{2}} = \frac{2}{3},$$

so that's

$$1 + \frac{1}{1 + \frac{2}{3}},$$

and

$$1 + \frac{2}{3} = \frac{5}{3},$$

so that's

$$1 + \frac{1}{\frac{5}{3}} = 1 + \frac{3}{5} = \frac{8}{5}.$$

Hence, the answer to the original question is $^8/_5$, which is the ratio of consecutive Fibonacci numbers. In fact, every step of the way were ratios of Fibonacci numbers.

$$1 + \frac{1}{1} = 2 = \frac{2}{1}$$

$$1 + \frac{1}{1 + \frac{1}{1}} = \frac{3}{2}$$

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} = \frac{5}{3}$$

This process always produces a ratio of consecutive Fibonacci numbers. Indeed, this process will continue forever since

$$1+\frac{1}{\frac{F_n}{F_{n-1}}}=1+\frac{F_{n-1}}{F_n}=\frac{F_n+F_{n-1}}{F_n}=\frac{F_{n+1}}{F_n}.$$

And that's the next consecutive pair of Fibonacci numbers. So, inductively, this process will generate Fibonacci fractions forever!

Proving That the Ratio of Fibonacci Numbers Converges on Phi

Recall from lecture 1 that the ratio of Fibonacci numbers seemed to get closer and closer to 1.618…. In other words, it looked like these fractions were converging to the golden ratio ϕ. Let's prove that now.

Imagine what would happen if you let that continued fraction go on forever. Let's call this infinitely long continued fraction x.

$$x=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}$$

But notice that x is equal to $1 + 1$ over the same infinitely long continued fraction.

That is, the infinitely long denominator is also x! In other words, you have the simple equation $x = 1 + {}^{1}/_{x}$. And when you multiply both sides by x, you get $x^2 = x + 1$, which is the key equation. And you know that the only positive number that solves this is ϕ. Hence, the ratio of Fibonacci numbers is guaranteed to get closer and closer to the number ϕ, as promised.

Proof by Contradiction

Any fraction that has an integer in the numerator and the denominator, such as $^2/_3$ or $^{17}/_{12}$ is called a rational number. The Fibonacci fractions—$^3/_2$, $^5/_3$, $^8/_5$, and so on—are also examples of rational numbers. And the numerator and denominator of these Fibonacci fractions are relatively prime, which means they have no common divisors, which guarantees that these fractions will be in lowest terms. And yet, even though the Fibonacci fractions are rational numbers, the number that they are getting closer and closer to is actually an irrational number. As will be proven, the number ϕ, which is equal to

$$\frac{1+\sqrt{5}}{2},$$

is irrational, which means that it cannot be written as a fraction where the numerator and denominator are integers.

Before the fact that ϕ is irrational is proven, let's first prove that the number $\sqrt{5}$ is irrational. This type of proof falls in the category of proving that something is impossible, called an impossibility proof. No matter how hard you try, you will never find two integers a and b so that $^a/_b = 5$.

The method for proving this is called proof by contradiction, and it goes like this. Suppose, to the contrary, that $\sqrt{5}$ is rational. Then, that means there exists integers a and b so that $\sqrt{5} = {}^a/_b$. That is, $^a/_b$ is a fraction in lowest terms. (All fractions can be written in lowest terms.) The goal is to reach a contradictory statement.

Let's square both sides of this equation. When you do this, you get the equation $5 = {}^{a^2}/_{b^2}$. That is, $a^2 = 5b^2$, which means that a^2 is a multiple of 5. It's 5 times some integer. But if a^2 is a multiple of 5, that can only happen if a is also a multiple of 5. That means that $a = 5k$ for some integer k. So, going back to the top equation, $5b^2 = a^2$, and $a = 5k$, so that's equal to the quantity $(5k)^2$, which is $25k^2$. So, $5b^2 = 25k^2$. Divide both sides by 5 to get that $b^2 = 5k^2$. That means that b^2 is a multiple of 5, and that means that b is a multiple of 5.

You've just shown that a is a multiple of 5 and b is a multiple of 5. But if that's the case, then the fraction $^a/_b$ can't be in lowest terms, so that's the contradiction. Hence, it is impossible to write $\sqrt{5}$ as a fraction, and therefore, $\sqrt{5}$ is irrational.

Proving That Phi Is Irrational

Now let's use this information to prove that ϕ is irrational. Again, let's do a proof by contradiction. Suppose that ϕ is rational—for example, $\phi = {}^a/_b$. (Let's not even require that $^a/_b$ is in lowest terms, but you could.) This means that $^{(1+\sqrt{5})}/_2 = {}^a/_b$. Remember, a and b are integers. If you multiply this by 2, it says that $1 + \sqrt{5} = {}^{(2a)}/_b$. And therefore, $\sqrt{5} = {}^{(2a)}/_b - 1 = {}^{(2a - b)}/_b$.

But since a and b are integers, the numerator $2a - b$ is an integer and the denominator b is an integer, which means that you just wrote $\sqrt{5}$ as an integer divided by an integer, and that's impossible—that's the contradiction—because you know that $\sqrt{5}$ is irrational. And thus, since $\sqrt{5}$ is irrational, you've shown that ϕ, the golden ratio, is irrational.

Phi Is the Most Irrational Number

In a very real sense, the golden ratio is the most irrational number—more irrational than $\sqrt{2}$ or pi (π) or any other irrational number you can name.

Recall that at the beginning of this lecture, you showed that ϕ can be represented as the following infinite continued fraction.

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1} + \ddots}}}$$

As it turns out, every irrational number has an infinite continued fraction. For example, π has a wild continued fraction.

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292} + \ddots}}}$$

And it goes on and on without pattern.

Infinite continued fractions allow you to approximate irrational numbers using rational numbers with very small denominators. Let's look at π again. You know that $\pi = 3.141592653589...$. Let's use its continued fraction to approximate it with fractions that have small denominators.

If you start from the top, you get that $\pi \approx 3$. But you can do better. You know that $\pi = 3.14159$.... So, if you look at the first 2 terms, you get that π is approximately $3 + {}^{1}/_{7}$.

That's ${}^{22}/_{7}$, which is 3.142857.... That's a little better. It agrees to 2 decimal places.

If you look at the first 3 terms, you get that π is approximately

$$3 + \frac{1}{(7 + \frac{1}{15})}.$$

his simplifies to ${}^{333}/_{106}$. That is 3.1415094..., which is accurate to 4 decimal places. And if you go to 4 terms, you get that π is approximately

$$3 + \frac{1}{(7 + \frac{1}{\frac{(15+1)}{1}})}$$

This simplifies to ${}^{355}/_{113}$, which is a pretty number because it has two 1s, two 3s, and 2 5s in it. And it starts 3.1415929..., so it's accurate to 6 decimal places.

Now let's turn to the continued fraction for ϕ, 1.618.... You know what its continued fraction representation looks like. The first few approximations of ϕ are the ratios of consecutive Fibonacci numbers.

${}^{1}/_{1} = 1.0$

${}^{2}/_{1} = 2.0$

${}^{3}/_{2} = 1.5$

${}^{5}/_{3} = 1.666...$

${}^{8}/_{5} = 1.60$

${}^{13}/_{8} = 1.625$

Even though the ratio of Fibonacci numbers is getting closer and closer to ϕ, the convergence is very slow. Notice that with π, by the fourth continued fraction, you had the estimate $^{355}/_{113}$, which was accurate to 6 decimal places. With ϕ, on the other hand, the fourth estimate is $^{8}/_{5} = 1.60$. That's accurate to only 1 decimal place. The reason the convergence is so slow is because the continued fraction for ϕ consists of all 1s. There are no big numbers in it, such as 15 or 292. As a result, ϕ is the irrational number with the slowest rate of convergence. In that sense, it can be said that the golden ratio ϕ is the most irrational number, because it's hard to get a good approximation of it using small denominators.

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9

A Golden Formula for Fibonacci

What is the 100th Fibonacci number? In theory, you can calculate it from the 99th and 98th Fibonacci numbers. But for these, you need the 97th and 96th Fibonacci numbers, and so on. Is there a formula that directly gives you the 100th Fibonacci number (or, more generally, the nth Fibonacci number) without having to recursively calculate all the previous ones?

Binet's Formula

The formula for the *n*th Fibonacci number is called Binet's formula, proved by the mathematician Jacques Philippe Marie Binet in 1843. As is often the case in mathematics, this formula was known more than a century prior to that by some mathematicians, including Daniel Bernoulli, Abraham de Moivre, and Leonhard Euler. On the other hand, Binet deserves some credit since he extended his formula to cover more sequences besides the Fibonacci numbers.

Here is Binet's formula. For $n \geq 0$, you can compute the *n*th Fibonacci number as follows:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Or, if you let

$$\phi = \frac{1+\sqrt{5}}{2}$$

and

$$\bar{\phi} = \frac{1-\sqrt{5}}{2},$$

then Binet's formula says that

$$F_n = \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}}.$$

Binet's formula can be used to provide algebraic proofs for many Fibonacci and Lucas identities.

Frankly, given that this formula uses the irrational numbers ϕ, $\bar{\phi}$ (phi-bar), and $\sqrt{5}$, it's amazing that this formula even produces integers, much less Fibonacci numbers. And yet, as you'll discover in this lecture, this formula does exactly that.

Before proving this, recall some of the properties of ϕ and $\bar{\phi}$.

$$\phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$$

$$\bar{\phi} = \frac{1-\sqrt{5}}{2} = -0.618\ldots$$

Adding these two fractions together, you get 1. (When you add the fractions, the $\sqrt{5}$s cancel, and you get $2/2$, which is 1.)

$$\frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2} = \frac{2}{2} = 1$$

Likewise, when you subtract the two fractions, you get $\sqrt{5}$. (The 1s cancel and you're left with $(2\sqrt{5})/2$, which is $\sqrt{5}$.)

$$\phi - \bar{\phi} = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5}$$

Numerically, this makes sense, too, since $1.618 + 0.618 = 2.236$, which is the beginning of $\sqrt{5}$.

Notice that when you multiply the two fractions, you get -1.

$$\phi\bar{\phi} = \frac{1+\sqrt{5}}{2}\,\frac{1-\sqrt{5}}{2} = \frac{1-5}{4} = \frac{-4}{4} = -1$$

And therefore, you can say that

$$\bar{\phi} = \frac{-1}{\phi}.$$

Finally, ϕ and $\bar{\phi}$ are solutions of the important key equation, $x^2 = x + 1$. Therefore,

$$\phi^2 = \phi + 1$$

and

$$\bar{\phi}^2 = \bar{\phi} + 1.$$

These make sense numerically, too, since 1.618^2 is about 2.618 and -0.618^2 is about 0.382.

Proving Binet's Formula

You are now ready to prove Binet's formula, which says that for $n \geq 0$,

$$F_n = \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}}.$$

Let's first observe that when $n = 0$, the formula says that

$$F_0 = \frac{\phi^0 - \bar{\phi}^0}{\sqrt{5}} = \frac{1 - 1}{\sqrt{5}} = 0.$$

This is correct for $n = 0$.

Likewise, when $n = 1$, Binet's formula says that

$$F_1 = \frac{\phi - \bar{\phi}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1.$$

This is also correct.

Notice that despite all the $\sqrt{5}$s, the formula is producing integers so far. Will it continue?

Let's suppose, inductively, that the formula works for two consecutive values—for example, n and $n - 1$. The goal is to prove that it will continue to work for the number $n + 1$. That is, you want to show that

$$F_{n+1} = \frac{\phi^{n+1} - \bar{\phi}^{n+1}}{\sqrt{5}}.$$

To prove this, notice that from the leapfrogging property, $F_{n+1} = F_n + F_{n-1}$, and from inductive assumptions,

$$F_n = \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}}$$

and

$$F_{n-1} = \frac{\phi^{n-1} - \bar{\phi}^{n-1}}{\sqrt{5}}.$$

Therefore, F_{n+1} will equal the sum of those two quantities.

$$F_{n+1} = \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}} + \frac{\phi^{n-1} - \bar{\phi}^{n-1}}{\sqrt{5}}$$

You can pull out a $1/\sqrt{5}$ from everything and put the ϕs together and the $\bar{\phi}$s together.

$$= \frac{1}{\sqrt{5}}[(\phi^n + \phi^{n-1}) - (\bar{\phi}^n + \bar{\phi}^{n-1})]$$

With that first ϕ expression, you can factor out a $\phi^{n-1}(\phi + 1)$ and do a similar operation for the second expression.

$$= \frac{1}{\sqrt{5}}[\phi^{n-1}(\phi + 1) - \bar{\phi}^{n-1}(\bar{\phi} + 1)]$$

But you know from the key equation that $\phi + 1 = \phi^2$ and $\bar{\phi} + 1 = \bar{\phi}^2$, so the equation becomes

$$= \frac{1}{\sqrt{5}}[\phi^{n-1}(\phi^2) - \bar{\phi}^{n-1}(\bar{\phi}^2)]$$

$$= \frac{1}{\sqrt{5}}[\phi^{n+1} - \bar{\phi}^{n+1}].$$

And this is exactly what you want. Thus, you've shown that Binet's formula works at the beginning, and it'll keep on working.

The same kind of induction argument can be used to prove that the Lucas numbers—2, 1, 3, 4, 7, 11, 18, 29, and so on—have an even simpler Binet formula (without $\sqrt{5}$s): $L_n = \phi^n + \bar{\phi}^n$.

Simplifying Binet's Formula

Notice that you can separate Binet's formula into two parts, so you can write it as

$$F_n = \frac{\phi^n}{\sqrt{5}} - \frac{\bar{\phi}^n}{\sqrt{5}}.$$

In practice, the second term will be very small. For instance, when $n = 7$,

$$\frac{\phi^7}{\sqrt{5}} - \frac{\bar{\phi}^7}{\sqrt{5}} = 12.9845 \ldots + 0.0154 \ldots = 13.$$

In fact, for every number $n \geq 0$, the second term,

$$\frac{\bar{\phi}^n}{\sqrt{5}} = \frac{(-0.618 \ldots)^n}{2.236 \ldots},$$

is guaranteed to have an absolute value less than 0.5. As a result, you can simplify Binet's formula even more by saying that F_n is always equal to $\phi^n/\sqrt{5}$ rounded to the nearest integer.

This offers another reason why the ratio of Fibonacci numbers gets closer and closer to ϕ. For large values of n, the nth Fibonacci number is almost exactly equal to $\phi^n/\sqrt{5}$. That is, F_n is approximately $\phi^n/\sqrt{5}$. And the next Fibonacci number, F_{n+1}, is almost exactly equal to $\phi^{n+1}/\sqrt{5}$. Hence,

$$\frac{F_{n+1}}{F_n} \approx \frac{\phi^{n+1}}{\sqrt{5}} \div \frac{\phi^n}{\sqrt{5}} = \phi.$$

In other words, this ratio is getting closer and closer to ϕ.

The same holds true for Lucas numbers. Since $L_n = \phi^n + \bar{\phi}^n$, you can say that for $n \geq 2$, L_n is equal to ϕ^n rounded to the nearest integer. For example, if you want to know the 20th Lucas number, the calculator says that $\phi^{20} = 15126.99993\ldots$, so therefore, $L_{20} = 15127$.

And if you want the 20th Fibonacci number, you just divide this number by $\sqrt{5}$, giving 6765.000029..., so therefore, $F_{20} = 6765$. Just like with Fibonacci numbers, this shows that the ratio of consecutive Lucas numbers gets closer and closer to ϕ.

The 100th Fibonacci and Lucas Numbers

When you compute ϕ^{100} on a calculator, it displays 7.920708398484e20, which means about $7.92... \times 10^{20}$. This means that L_{100} is a 21-digit number, approximately 800 quintillion, but it doesn't give the number exactly. Dividing this by $\sqrt{5}$, the 100th Fibonacci number is about 3.54×10^{20}, which is around 350 quintillion. The exact number turns out to be 354,224,848,179,261,915,075.

Binet's formula shows that you can find the nth Fibonacci using powers of ϕ. Now let's reverse the situation and show that you can find a simple expression for the powers of ϕ using Fibonacci numbers. It's all based on the key equation: $\phi^2 = \phi + 1$.

If you multiply this equation by ϕ, you get that $\phi^3 = \phi^2 + \phi$, but $\phi^2 = \phi + 1$, so that means that $\phi^3 = (\phi + 1) + \phi$, and that's equal to $2\phi + 1$. If you multiply this by ϕ again, you get $\phi^4 = 2\phi^2 + \phi$. Use the key equation and replace ϕ^2 with $\phi + 1$ and you have $2(\phi + 1) + \phi$, which is $3\phi + 2$. Multiplying this by ϕ again, you get that $\phi^5 = 3\phi^2 + 2\phi$, but $\phi^2 = \phi + 1$, so that's $3(\phi + 1) + 2\phi$, which is $5\phi + 3$.

The next power will be $\phi^6 = 8\phi + 5$. That's $F_6\phi + F_5$. The general pattern looks like this: $\phi^n = F_n\phi + F_{n-1}$.

That's certainly the pattern that's seen at the beginning, and you can use induction to show that this pattern will persist forever. That is, for all values of n, $\phi^n = F_n\phi + F_{n-1}$.

Notice that the only property of ϕ that you used was the key equation, so this theorem will also apply to $\bar{\phi}$. That is, it'll also be the case that $\bar{\phi}^n = F_n\bar{\phi} + F_{n-1}$.

Amazingly, this leads to a superfast proof of Binet's formula. All you have to do is subtract those two equations, and you end up with Binet's formula.

$$\begin{aligned}\phi^n - \bar{\phi}^n &= (F_n\phi + F_{n-1}) - (F_n\bar{\phi} + F_{n-1}) \\ &= F_n(\phi - \bar{\phi}) = F_n\sqrt{5} \\ F_n &= \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}}\end{aligned}$$

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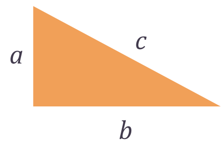
10

The Golden Ratio and Geometry

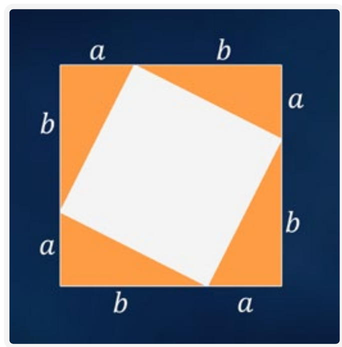
Johannes Kepler was a great German mathematician and astronomer of the 16th century whose laws of planetary motion inspired the work of Isaac Newton. Kepler believed that the physical laws of nature could be described using geometry. He said, “Where there is matter, there is geometry.” He was also a big fan of the golden ratio. This lecture explores the geometric properties of the golden ratio.

The Pythagorean Theorem

The Pythagorean theorem says that for any right triangle with side lengths a and b and hypotenuse c, it's always the case that $a^2 + b^2 = c^2$.



There are hundreds of different proofs of the Pythagorean theorem, but here's a fun and simple one. Imagine making 4 copies of a right triangle in such a way as to create a giant square with side lengths $a + b$. What's the area of the entire region?



Answer 1: It's $(a + b)^2 = a^2 + 2ab + b^2$.

Answer 2: It's the area of 4 triangles plus the area of the square in the middle. Each triangle has an area of $\frac{1}{2}ab$, and the area of the square in the middle is c^2, so the total area would be $4(\frac{1}{2}ab) + c^2 = 2ab + c^2$.

Since both answers are correct, they must be equal. That is, $a^2 + 2ab + b^2 = 2ab + c^2$. When you subtract $2ab$ from both sides, you get $a^2 + b^2 = c^2$, as desired.

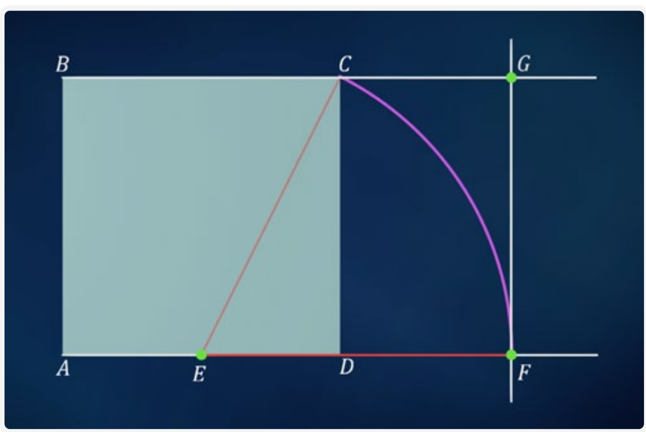
Kepler's Triangle

Kepler asked whether it was possible to have a right triangle where the sides were constant multiples of each other. Let's say the shorter side has length 1 and we use a constant multiple of a. Then, the triangle would look like this: 1, a, and a^2. By Pythagoras's theorem, it must be the case that $1^2 + a^2 = (a^2)^2$. In other words, $1 + a^2 = a^4$.

If I let $x = a^2$, this equation becomes $1 + x = x^2$, which is our good friend the key equation. And we know that the only positive solution to the key equation is when x is the golden ratio ϕ. And since $a^2 = \phi$, then the only positive solution to Kepler's problem is when a is the $\sqrt{\phi}$, that's $\sqrt{1.618}$, approximately 1.272. This triangle is known as Kepler's triangle.

The Golden Rectangle

The ancient Greeks, such as Pythagoras, Euclid, and Archimedes, were masters of geometry. One of the mathematical challenges they set for themselves was to create a golden rectangle starting from a 1-by-1 square. Here's how it was done. Label the square $ABCD$, find the midpoint of AD, and call it E. Then, with a compass, put one end on E and the other on C. Then, draw the arc of a circle, centered at E, and see where it hits the line going through AD and call that point F. Finally, draw the vertical line through F, see where it hits the line through BC, and call that new point G (as in "golden"). The claim is that the rectangle $ABGF$ is a golden rectangle. Let's prove this.



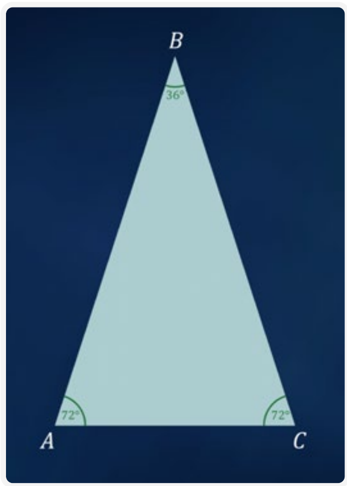
You know that the height of the rectangle is 1. Let's show that the width of the rectangle (that is, the length of AF) is exactly ϕ.

From the original square, you know that the distance from A to E is ½. What's the distance from E to F? From the compass, it's the same as the distance from E to C. And if you look at the right triangle EDC, you can use the Pythagorean theorem to see that $EC^2 = ED^2 + DC^2 = (½)^2 + 1^2 = 5/4$. Therefore, $EF = EC = \sqrt{5}/2$. Hence, the length of line AF is equal to $AF = AE + E$. And $AE = ½$ and $EF = \sqrt{5}/2$, and when you add those, you get $(1+\sqrt{5})/2$, which is ϕ. That is, the length of AF is ϕ. Hence, the rectangle you've constructed is indeed a golden one.

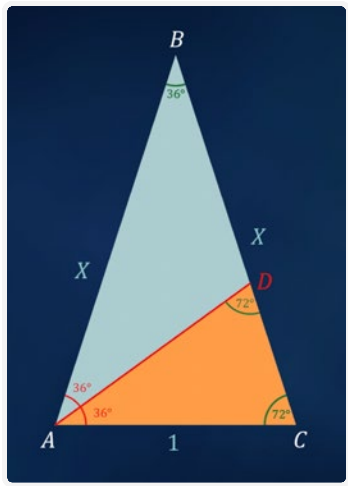
Golden Triangles and Golden Gnomons

The golden triangle is an isosceles triangle where the small angle is 36° and the larger two angles are 72°. Notice that the sum of the angles is 72° + 72° + 36° = 180°, which is true of all triangles.

An isosceles triangle has two equal angles and two equal sides, and if one condition holds, so must the other. Here, you have two 72° angles, so their opposite sides must have the same length. What is the ratio of the long side to the short side? In other words, if the short side has length 1, then what is the length of the long side? Let's call the length of the long side X. The goal is to determine X using the simplest tools from geometry.



Let's draw a line from the point *A* that cuts the 72° angle in half. (This is called the angle bisector.) Let's say that it intersects the other side of the triangle at the point *D*.



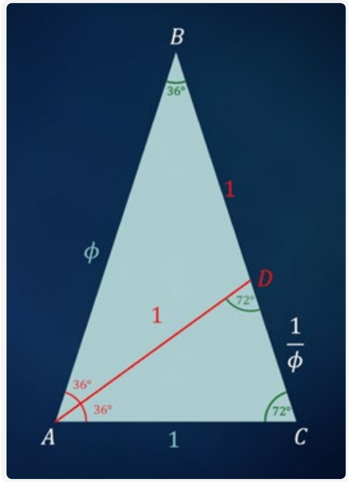
Triangle *ADC* has a 36° angle and a 72° angle. Since the angles of the triangle must add to 180°, this means that angle *ADC* must be 72°.

But notice that triangle *DAC* has two 72° angles, which means that triangle *DAC* is isosceles, and therefore, the length of side *AD* must be the same as the length of side *AC*, which is 1. Thus, the length of *AD* is 1.

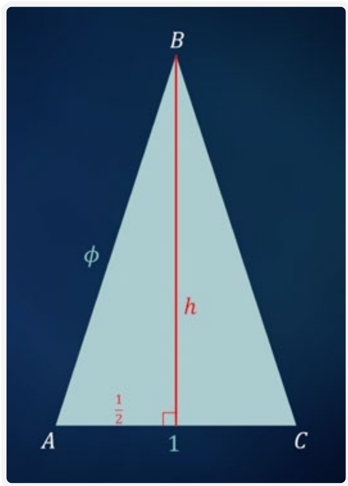
Also notice that triangle *BDA* has two angles of 36°, so *BDA* is also isosceles. This means that the length of *BD* must equal the length of *AD*, which is 1. Therefore, the length of *BD* is also 1. And since side *BC* has length *X*, this means that the length of *DC* must be $X - 1$.

Finally, notice that triangle *DAC* is an isosceles triangle with angles 72°, 72°, and 36°, just like the original triangle, *ABC*. Triangles *DAC* and *ABC* are said to be similar, which means that one is just a scaled version of the other and that their side lengths are proportional. The big triangle, *ABC*, has lengths *X*, *X*, and 1. The smaller triangle, *DAC*, has side lengths 1, 1, and $X - 1$.

Since the triangles are similar, the ratio of the long length to the short length must be the same in both triangles. Therefore, $X/1 = 1/(X-1)$. In other words, $X = 1/(X-1)$, and when you multiply both sides by $X - 1$, this says that $X(X-1) = 1$, which is to say that $X^2 - X = 1$, and therefore, $X^2 = X + 1$, which is the key equation. And you know that the only positive solution to the key equation is ϕ, and therefore, $X = \phi$; thus, the ratio of the long side to the short side is the golden ratio, approximately 1.618. In fact, you can even replace $\phi - 1$ with $1/\phi$, and you can see why this is called the golden triangle since the long sides are ϕ times longer than the short side.

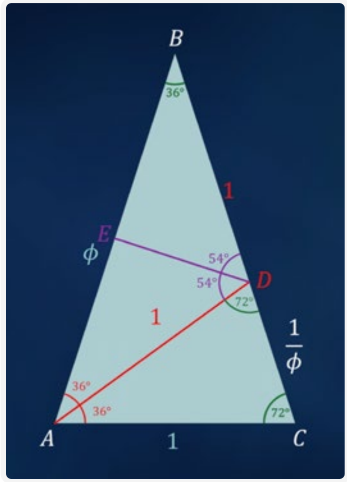


Now that you know the length of each side, you can easily compute other quantities. For instance, you can determine the height of the triangle from the Pythagorean theorem. Since $\phi^2 = h^2 + (½)^2$, $h = \sqrt{\phi^2 - ¼}$, which is approximately 1.53. And since the area of the triangle is ½ base × height and the base is 1, you get an area of about 0.77.

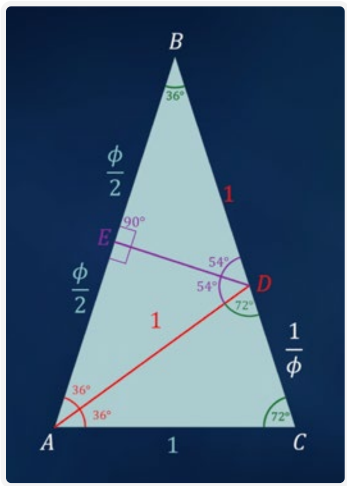


There's another piece of information revealed by this triangle. What is angle *ADB*? Since angle *ADC* is 72°, then angle *ADB* must be 108°. Let's draw the angle bisector to split that 108° angle into two 54° angles intersecting side *AB* at the point *E*.

Notice that triangle *BDE* has angles of 36° and 54°, which add up to 90°. Thus, for triangle *BDE* to add up to 180°, angle *BED* must also be 90°, so it's a right angle—as is angle *AED*. In fact, triangles *BDE* and *ADE* can be shown to be congruent triangles, and therefore, *BE* and *AE* will have the same length. And since the length of side *BA* is ϕ, that means that *BE* and *AE* both have length $\phi/2$.



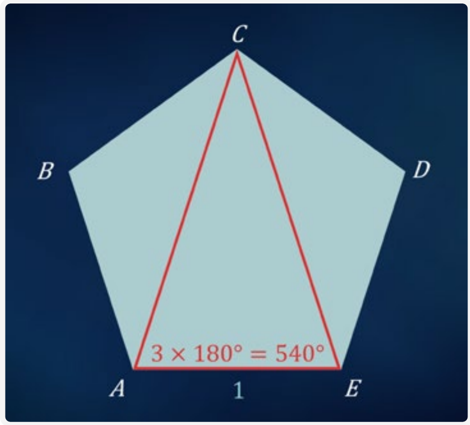
In a right triangle, such as *BED*, the sine of an angle is the length of the opposite side divided by the length of the hypotenuse. Thus, if you look at angle *BDE*, you see that the sine of 54° is $\phi/2$, and therefore, $\phi = 2 \sin 54°$.



By the same argument, you can also find ϕ using the cosine of angle *EBD*. The cosine of an angle in a right triangle is the length of the adjacent, non-hypotenuse side divided by the length of the hypotenuse. So, you have the cosine of 36° is $\phi/2$, and therefore, $\phi = 2 \cos 36°$.

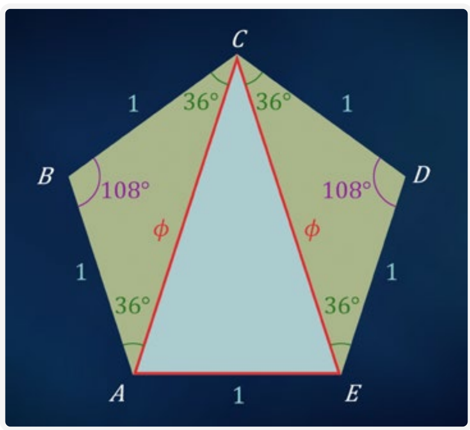
Finally, notice that triangle *ABD* is also an isosceles triangle with angles 36°, 36°, and 108°. Triangles with these angles are called golden gnomons because their lengths have golden proportions. For example, triangle *ABD* has side lengths 1, 1, and ϕ. Similar to the previous calculation, you can use the Pythagorean theorem to compute the length of *DE* and subsequently the area of the golden gnomon.

Pentagons and Pentagrams

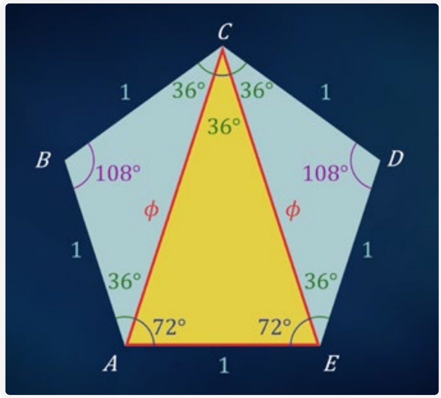


You've now explored golden objects with 3 sides and 4 sides, but perhaps the most golden object of all contains 5 sides: a regular pentagon, which has sides of length 1. In any pentagon, the sum of the interior angles is always 540°. To see this, notice that all of the interior angles come from the angles of 3 triangles, so the angles will sum to 3 × 180°, which is 540°.

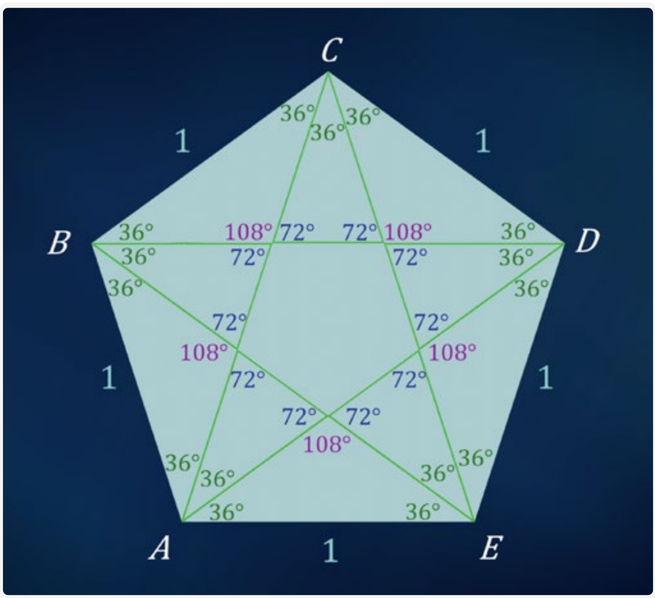
Since the angles sum to 540°, then each big angle, such as angles B and D, will have a measure of $540/5$, which is 108°. And since sides AB and BC both have length 1, triangle ABC must be isosceles, so in order for its angles to add to 180°, the other two angles must both be 36°. The same goes for triangle CDE. But this means that triangles ABC and CDE are golden gnomons, and since their short lengths are 1, their long length is equal to ϕ. In other words, in a regular pentagon with sides of length 1, the lengths of all the diagonals (such as AC and CE) must be the golden ratio ϕ.



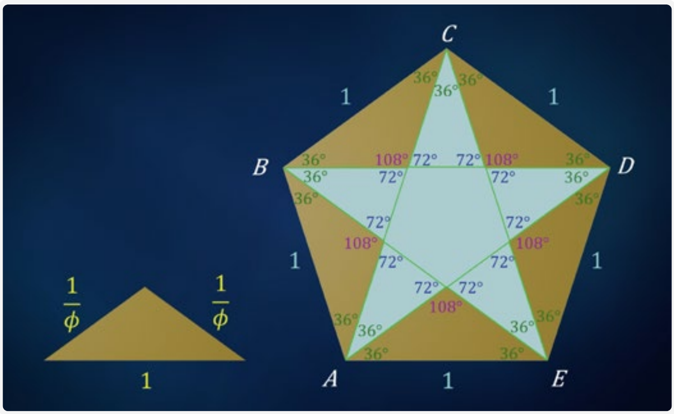
The triangle ACE is the golden triangle! Notice that triangle *ACE* has lengths ϕ, ϕ, and 1, and its angles are 72°, 72°, and 36°. It's as if the regular pentagon is a monument dedicated to the golden ratio ϕ!



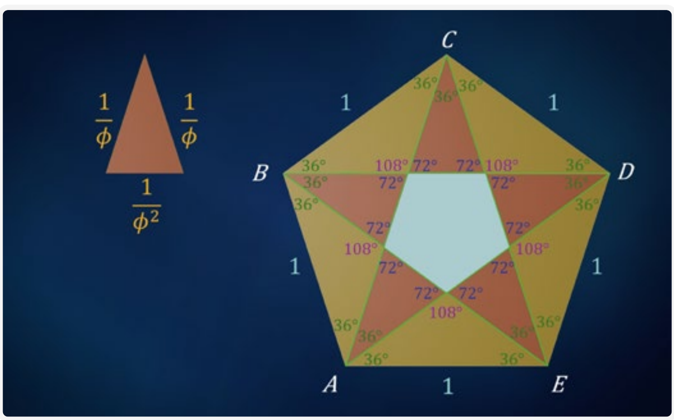
Once you add the other 3 diagonals to the pentagon, you get a bunch of triangles where all the angles are either 36°, 72°, or 108°. This results in a 5-pointed star, which mathematicians call a pentagram.



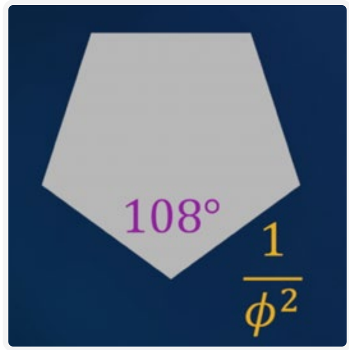
Between the pentagram and the pentagon are 5 triangles with angles 36°, 36°, and 108°, which are golden gnomons, and thus they have golden proportions: $1/\phi$, $1/\phi$, and 1. Each golden gnomon has a long side of length 1 and therefore two short sides of length $1/\phi$.



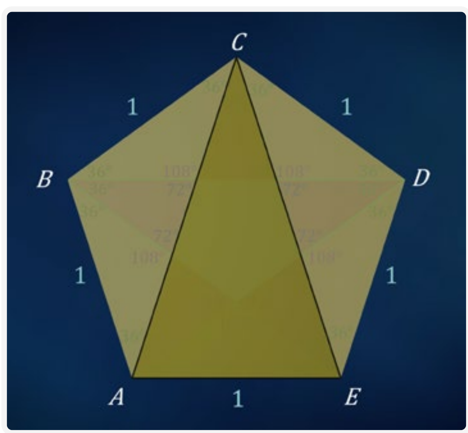
The pentagram consists of 5 golden triangles and a regular pentagon in the middle. Each golden triangle has two long sides of length $1/\phi$ and a short side of length $1/\phi^2$.

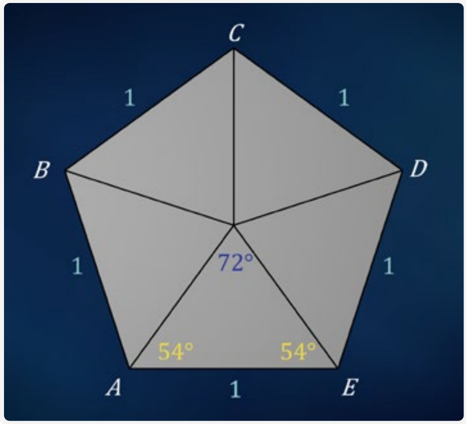


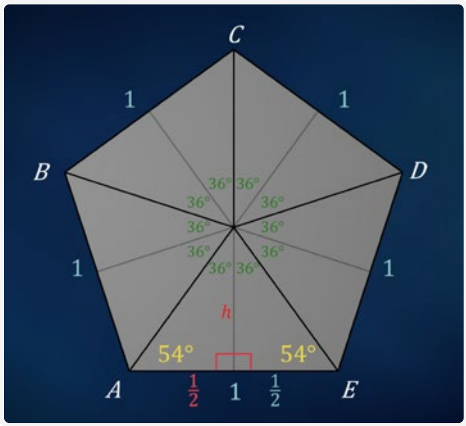
The pentagon in the middle is a regular pentagon since all the angles are equal to 108° and all the lengths are equal to $^1/_{\phi^2}$. This means that the middle pentagon is a scaled-down version of the original pentagon, where each side is scaled down by a factor of ϕ^2. This means that the pentagon's perimeter is scaled down by a factor of ϕ^2. So, since the big pentagon has a perimeter of 5, then the smaller pentagon has a perimeter of $^5/_{\phi^2}$. And the area of the smaller pentagon will be reduced by a factor of $(\phi^2)^2$, or ϕ^4, which is about 6.85. Thus, the inner pentagon is about $^1/_7$th the area of the big pentagon.



And notice that the big pentagon can be broken into two large golden gnomons and one large golden triangle. And since you know how to compute the areas of golden gnomons and golden triangles, you can use that to determine the area of the regular pentagon.







But there's a quicker, more elegant way. You can break up the pentagon into 5 isosceles triangles where the angles in the center are 72° ($5 \times 72° = 360°$) and the other angles, because they're isosceles triangles, are 54° ($54° + 54° + 72° = 180°$).

You can then split each of the triangles in half to create 10 congruent right triangles with angles measuring 36°, 54°, and 90°. The area of each triangle is ½ base × height. Each base has a length of ½, and you can use trigonometry—specifically the tangent function—to find the height, h.

The tangent of the 54° angle is the length of the opposite side divided by the length of the adjacent side. That is, the tangent of 54° is h divided by ½, which is $2h$. So, $h = (\tan 54°)/2$, so the area of each little triangle is $½ \times ½ \times (\tan 54°)/2$. That's equal to $(\tan 54°)/8$.

Then, you can multiply the area of the little triangle by 10 to get the area of the pentagon. Thus, the area of the pentagon is $^{10}/_{8} \times \tan 54°$. A calculator indicates that tan 54° is about 1.376, so the area of the regular unit pentagon is about 1.72.

This is the area when every side has length 1. If, instead, every side has length *s*, you simply multiply this by the number s^2.

The Fibonacci Association is an organization devoted to studying beautiful properties of the Fibonacci numbers, the golden ratio, and other fascinating number sequences.

READING

Koshy, Thomas. *Fibonacci and Lucas Numbers with Applications*. Hoboken, NJ: Wiley, 2017.

Meisner, Gary B., and Rafael Araujo. *The Golden Ratio: The Divine Beauty of Mathematics*. New York: Race Point Publishing, 2018.

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11

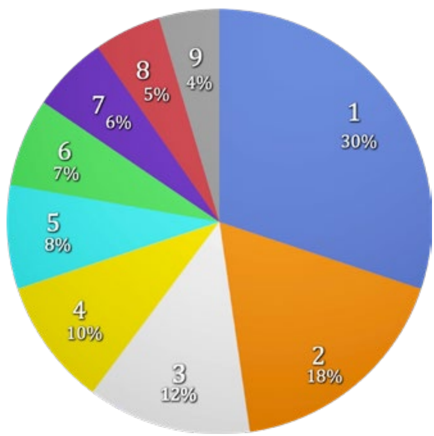
The Golden Ratio and Benford's Law

How many of the first 10,000 Fibonacci numbers begin with the digit 1? You might assume that it would be about $^1/_9$th (about 11%) of them since the first digit can be any number between 1 and 9. However, it turns out that almost exactly 30% of the Fibonacci numbers begin with the digit 1. In second place, about 18% of the Fibonacci numbers begin with the digit 2, and the percentages steadily decrease through the digit 9, which occurs less than 5% of the time. And if you looked instead at the first 10,000 powers of ϕ, the leading digits occur with the same frequency! This phenomenon is known as Benford's law, which—like the Fibonacci numbers and the golden ratio—shows up all over mathematics and the real world.

Benford's Law

Here are the complete tables showing the leading digits of the first 10,000 Fibonacci numbers as well as the first 10,000 powers of ϕ, rounded to the nearest decimal.

First 10,000 Fibonacci numbers		First 10,000 powers of ϕ	
Leading Digit	Frequency	Leading Digit	Frequency
1	30.1%	1	30.1%
2	17.6%	2	17.6%
3	12.5%	3	12.5%
4	9.7%	4	9.7%
5	7.9%	5	7.9%
6	6.7%	6	6.7%
7	5.8%	7	5.8%
8	5.1%	8	5.1%
9	4.6%	9	4.6%



You can visualize this by the pie chart shown here, where numbers are rounded to the nearest percent.

It turns out that these numbers are all coming from logarithms. The first number on the chart (rounded to 3 places) is the logarithm of 2. That is, $\log 2 = 0.301\ldots$. Remember, the logarithm of any number is the power of 10 that gives that number. For example, $\log 100 = 2$ since 10^2 is 100. And when you say that the log of 2 is 0.301, that means that when you raise 10 to the 0.301 power, you get 2. That is, $10^{\log 2} = 10^{0.301\ldots} = 2$.

In other words, the probability that the leading digit is $\leq d$ is the log of $(d + 1)$. For example, the probability that the leading digit is less than or equal to 6 is the log of 7. The probability that the leading digit equals 6 would be the probability that it's less than or equal to 6 minus the probability that it's less than or equal to 5. Thus, the probability of beginning with 6 would be $\log 7 - \log 6$.

In general, Benford's law states that the probability (or the long-run frequency) of starting with digit d is the log of $(d + 1)$ minus the log of d. For example, the probability of starting with the digit 2 is $\log 3 - \log 2 = 0.176$.

Leading Digit	Frequency
1	30.1%
2	17.6%
3	12.5%
4	9.7%
5	7.9%
6	6.7%
7	5.8%
8	5.1%
9	4.6%

Notice that these probabilities add up to 1 since when you add the numbers in the last column, everything cancels—except for log 10. And the log of 10 is 1, so the probabilities add up to the log of 10, which equals 1.

Why should these percentages have anything to do with logarithms? There's something interesting that all numbers that begin with 1 have in common, and it has to do with logarithms.

The Mantissa

For a real number to begin with the digit 1, there are a bunch of possibilities. Either the number is between 1 and 2 (technically, 1.99999…), between 10 and 20, between 100 and 200, or between 1000 and 2000, and so on. What can be said about the logarithms of these numbers?

For numbers between 1 and 2, the logarithm can be as small as the log of 1, which is 0, and as big as the log of 2, which is 0.301. So, for numbers between 1 and 2, the logarithm is guaranteed to be between 0 and 0.301. For numbers between 10 and 20, the logarithm can be as small as the log of 10, which 1, and the log of 20, which is 1.301. For numbers between 100 and 200, the logarithm ranges from the log of 100, which is 2, and the log of 200, which is 2.301. For numbers between 1000 and 2000, the logarithm ranges from the log of 1000, which is 3, and the log of 2000, which is 3.301.

Why do you keep getting this 0.301 term? It's because of one of the fundamental rules of logarithms, which says that the log of the product is the sum of the logs. That is,

$$\log(ab) = \log a + \log b.$$

For example,

$$\begin{aligned} \log 2000 &= \log(1000 \times 2) \\ &= \log 1000 + \log 2 \\ &= 3 + 0.301 \\ &= 3.301. \end{aligned}$$

In other words, for any number that begins with 1, when you take its logarithm, the stuff after the decimal point, called the mantissa, has to be between 0 and 0.301. For example, the number 123456 begins with 1, and the log of 123456 is 5.091…, so the mantissa of that number is 0.091, and that's between 0 and 0.301, as it should be. Conversely, any number whose logarithm has a mantissa of 0.091, whether it came from 1.091 or 5.091 or 2358.091, has to begin with the digit 1.

What about numbers that begin with the digit 2? For a number between 2 and 3 (again, technically, 2.999...), the logarithm ranges from the log of 2, which is 0.301, to the log of 3, which is 0.477. For numbers between 20 and 30, the logarithm goes from the log of 20, which is 1.301, to the log of 30, which is 1.477. For numbers between 200 and 300, the logarithm goes from the log of 200, which is 2.301, to the log of 300, which is 2.477, for the same reason as before: $\log 300 = \log 100 \times 3 = \log 100 + \log 3 = 2 + 0.477 = 2.477$.

In other words, every number that begins with 2, when you take its logarithm, the mantissa must be between the log of 2 (0.301) and the log of 3 (0.477).

In general, for any number that begins with digit d, after taking its logarithm, the mantissa is guaranteed to be between the log of d and the log of $(d + 1)$.

Leading Digit	Frequency	Mantissa
1	30.1%	0 to 0.301
2	17.6%	0.301 to 0.477
3	12.5%	0.477 to 0.602
4	9.7%	0.602 to 0.699
5	7.9%	0.699 to 0.778
6	6.7%	0.778 to 0.845
7	5.8%	0.845 to 0.903
8	5.1%	0.903 to 0.954
9	4.6%	0.954 to 0.999

Let's look at the first 10 powers of ϕ, along with their logs and mantissas, starting with ϕ^1, or 1.618…. Its logarithm is about 0.209…. More exactly, the log of ϕ is 0.20898764…, which can be shown to be an irrational number, meaning that it cannot be expressed as a simple fraction.

Notice that its mantissa, 0.209, is between 0 and 0.301, and indeed, that makes sense because ϕ begins with the digit 1. Here are the first 10 powers of phi, along with their logs and the mantissas of their logs. Everything is rounded to 3 decimal points, and the mantissas are bolded.

$\phi^1 = 1.618;\quad \log \phi = 0.\mathbf{209}$

$\phi^2 = 2.618;\quad \log \phi^2 = 0.\mathbf{418}$

$\phi^3 = 4.236;\quad \log \phi^3 = 0.\mathbf{627}$

$\phi^4 = 6.854;\quad \log \phi^4 = 0.\mathbf{836}$

$\phi^5 = 11.090;\quad \log \phi^5 = 1.\mathbf{045}$

$\phi^6 = 17.944;\quad \log \phi^6 = 1.\mathbf{254}$

$\phi^7 = 29.034;\quad \log \phi^7 = 1.\mathbf{463}$

$\phi^8 = 46.978;\quad \log \phi^8 = 1.\mathbf{672}$

$\phi^9 = 76.013;\quad \log \phi^9 = 1.\mathbf{881}$

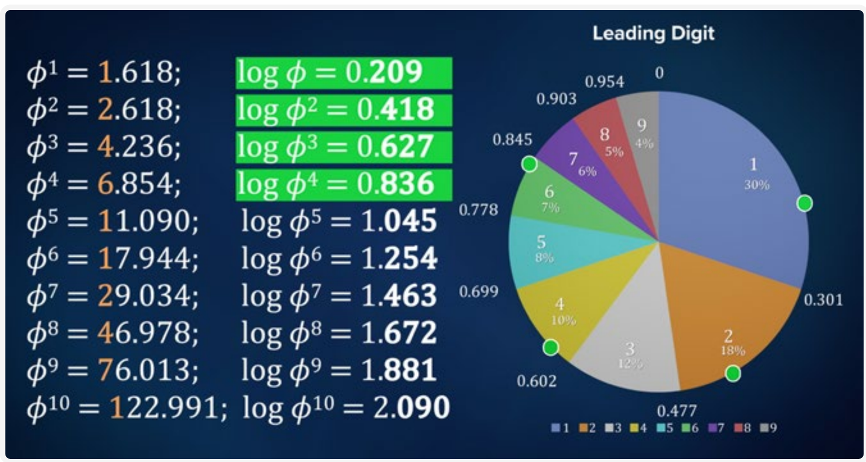
$\phi^{10} = 122.991;\quad \log \phi^{10} = 2.\mathbf{090}$

Leading Digit	Mantissa
1	0 to 0.301
2	0.301 to 0.477
3	0.477 to 0.602
4	0.602 to 0.699
5	0.699 to 0.778
6	0.778 to 0.845
7	0.845 to 0.903
8	0.903 to 0.954
9	0.954 to 0.999

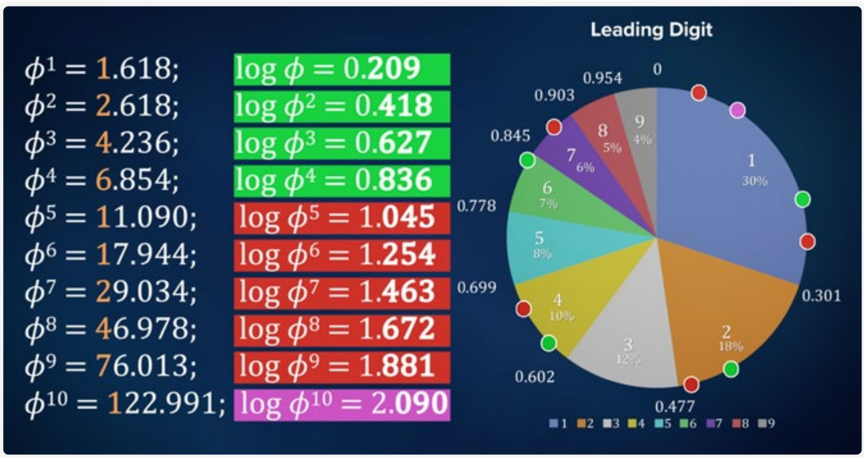
The Logarithm Law of Exponents

Notice that the logarithms follow a very regular pattern: They're all multiples of 0.209. That's because of the logarithm law of exponents, which says that for any positive number b, the log of b^N is $N \times$ the log of b. In particular, with base ϕ, the log of ϕ^N is $N \log \phi$, which is $N \times 0.209$. Thus, every row is guaranteed to be a multiple of 0.209. Another way to say this is that you can find the value of the log of ϕ^N by simply adding 0.209 to the previous value.

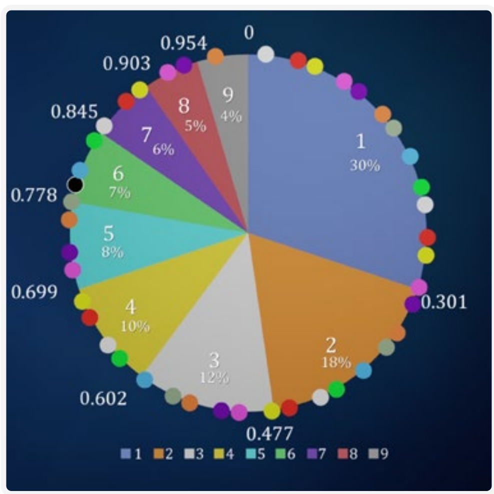
Let's plot the first 4 mantissas—0.209, 0.418, 0.627, and 0.836—on the rim of the circle. Notice how evenly spaced they are.



For the next mantissa, you go around the circle, throwing away the 1, to reach 0.045. You continue to add 0.209 for the next 4 points. And when you add the next 0.209, you go around the circle again, landing at 0.090.



Look what happens as you plot more and more of these dots.



Note that because the log of ϕ 0.20898... is irrational, no two points will ever be at exactly the same spot. So, as you add more and more points to the circle, these dots will uniformly fill the entire rim of the circle. So, in the long run, you expect to see 30.1% of the points to be in the first region, which represents powers of ϕ with first digit 1. The percentage of points with first digit 2 should be 47.7% − 30.1%, or 17.6%, and so on, exactly as Benford's law predicts.

Note that there was nothing particularly special about powers of ϕ. You could have used powers of any number b as long as the log of b is irrational. But since Benford's law holds for powers of ϕ, then that implies that it works for the Lucas numbers since, except for the first few values of n, the nth Lucas number is almost exactly equal to ϕ^n and thus should have the same first digit.

What happens if you multiply every ϕ^n term by some constant c? By the usual log rule,

$$\log(c\phi^n) = \log c + \log \phi^n.$$

In other words, the mantissas all rotate by the log of c. And since the mantissas of the powers of ϕ are uniformly distributed along the rim of the circle, then it'll remain true even after you rotate it by any fixed amount. Consequently, any sequence of the form $c\phi^n$ will also obey Benford's law.

> **In recent years, Benford's law has developed somewhat of a cult following, and people look for it everywhere: in nature, in economics, in political science, and just about wherever numerical data spans several orders of magnitude.**

But you know from Binet's formula that, except for the first few terms, the Fibonacci numbers are almost exactly equal to $\phi^n(1/\sqrt{5})$. That's the nth Fibonacci number. And so, with a large collection of Fibonacci numbers, they must also obey Benford's law.

Nonmathematical Situations

A curious aspect of Benford's law that's not particularly well understood is that it even seems to apply in many nonmathematical situations. For instance, if you open an almanac and it lists the lengths of all the rivers in the world, then regardless of whether the rivers are measured in meters, miles, or footsteps, the first digits should be very close to what Benford's law predicts. That is, about 30% of them should begin with 1, and so on.

Or if you ask a large group of people to give you the first digit of their street address, about 30% of them will begin with 1, about 18% will begin with 2, about 12% will begin with 3, and so on. Notice that about 60% of the addresses will begin with the 3 smallest numbers, whereas only about 15% will begin with the 3 largest numbers (7, 8, and 9 combined). In other words, the ratio of addresses that begin with a small number versus a big number is practically 4 to 1!

Some people say that this makes sense since when people are numbering buildings, they have to go through the smaller digits first (the 100s, 200s, and 300s) before they get to the larger numbers, if they ever get there. However, the same phenomenon occurs if you ask people to think of their home address and double it or triple it and then look at the first digit.

Benford's law seems to apply to just about any collection of numbers where the size of the numbers spans several orders of magnitude. For instance, river lengths could be 1 digit, 2 digits, 3 digits, or 4 digits long. This is also the case for most street addresses in the US, which can vary from 1 digit to about 5 digits—or the populations of every US county.

But it won't apply to the scores on a math exam since almost all the percentages will be 2-digit numbers. Or if you ask everyone in a room for their height in inches, then everyone should have a 2-digit number, and you certainly wouldn't expect to see 30% of them beginning with the number 1. In fact, you wouldn't see anyone with a height that begins with 1 unless you know people that are either less than 20 inches tall or more than 100 inches tall!

A similar issue arose in a recent presidential election where someone noticed that in one set of precincts, one candidate's votes per precinct did not mesh up with Benford's law and wondered if that was grounds for fraud. But since all the precincts were practically the same size, Benford's law did not apply. When applied to all the counties in a state or country, Benford's law usually shows up, but there can be exceptions.

Another place where numbers of various different lengths are seen are on tax returns, and according to Benford's law, about 30% of the numbers on the return should begin with 1, about 18% should begin with 2, and so on. If the numbers differ wildly from these percentages, as could happen when the numbers are manipulated by humans, then the agency might not know where the errors are or even if the return is fraudulent, but they'd have the impression that the tax return smells funny and might give it closer inspection.

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12

Fibonacci and the Golden Ratio Everywhere

This lecture explores some of the reasons behind the frequent appearance of the Fibonacci numbers and the golden ratio in nature, along with its use and misuse in other areas, including art and architecture.

Nature Prefers Irrational Numbers

In addition to possessing fascinating mathematical properties, the Fibonacci numbers and the golden ratio seem to be all around you in the natural world. For example, the number of petals on a flower tends to be a Fibonacci number. For instance, lilies and irises have 3 petals. Buttercups, larkspurs, and columbines have 5 petals. Some delphiniums have 8 petals. Corn marigolds and black-eyed Susans have 13 petals. Some asters have 21 petals. Some daisies can be found with 34, 55, or 89 petals!

Other numbers are possible, but they tend to be rarer, such as 4-leaf clovers. And in some cases, the number of petals will vary, but the average is often a Fibonacci number. Or if you look at the number of spirals on a sunflower, pinecone, or pineapple, you tend to see a Fibonacci number of spirals in one direction along with a Fibonacci number of spirals in the opposite direction.

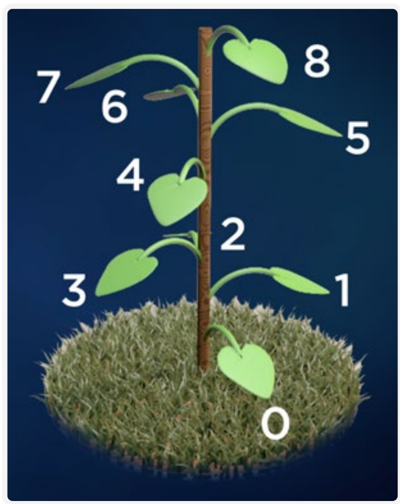
The Fibonacci numbers and the golden ratio appear in nature so much that they're sometimes referred to as "nature's numbers." The main reason Fibonacci numbers and the golden ratio are seen in nature stems from the fact that ϕ, which is $(1+\sqrt{5})/2$ or 1.618…, is an irrational number. Additionally, as was shown in lecture 8, it is the most irrational number. And even though you might find rational numbers such as ½ or $2/3$ to be very natural, when it comes to plants and flowers, nature tends to prefer irrational numbers.

Suppose that as a plant or new tree is growing, it sprouts a leaf every 90°, or ¼ of a circle, so after the first 4 leaves appear, the 5th leaf would be directly above the 1st leaf, the 6th leaf would be directly above the 2nd leaf, and so on. And that would not be good for getting sunshine and water to the leaves at the bottom. To avoid this problem of one leaf being directly over another, you need the angle to be an irrational number when divided by 360°. But which irrational number? There are lots to choose from. The more irrational the better! And as you discovered in lecture 8, the most irrational number is ϕ, which is 1.618…, or, equivalently, $^1/_\phi$, which is 0.618….

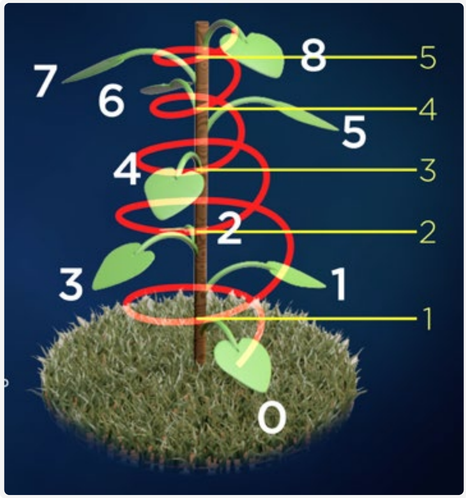
Suppose instead of taking ¼ of a circle, you took $^1/_\phi$, or about 0.618, of a circle. When you multiply that by 360°, that's about 222.5°. Sometimes mathematicians like to use angles that are less than 180°, so instead let's use the complement of that angle, which is 137.5° (360° − 222.5° = 137.5°). And that happens to be $^{360^\circ}/_{\phi^2}$.

These are called the golden angles, which are often seen between consecutive leaves on plants where the leaves grow in a spiral pattern. It's been observed that many plants sprout a new leaf or petal at golden angles so that they get the best photosynthetic exposure to the elements. And for this reason, the golden ratio and the Fibonacci numbers appear in nature surprisingly often.

One way that botanists classify plants is by performing two measurements. As you count the leaves from the bottom, let q be the number of leaves until one leaf is directly above the one on the bottom (not counting the initial leaf). For instance, in this plant, that number would be 8. If you call the bottom leaf 0, the leaf that is directly above it is leaf 8.



Then, count the number of complete turns that you make during that cycle. Let's call this number p. Here, that number would be 5.



The ratio of p/q is the called the divergence of the plant. In this example, the angle between consecutive leaves is 222.5°. So, after 8 leaves, you've completed $8 \times 222.5° = 1800°$ turns. That's $5 \times 360°$, so it makes sense that we would complete exactly 5 rotations.

Consequently, for many plants, the divergence is the ratio of Fibonacci numbers that are 1 or 2 apart (depending on which golden angle was used). Here's a list of some plants and their divergences. (Again, not all plants have this feature, but a surprising number of them do.)

Elm, linden, lime, some common grasses	$\frac{1}{2}$
Beech, hazel, blackberry, sedges, some grasses	$\frac{1}{3}$
Oak, cherry, apple, holly, plum	$\frac{2}{5}$
Poplar, rose, pear, willow	$\frac{3}{8}$
Almond, pussy willow, leeks	$\frac{5}{13}$

The study of leaf arrangements, known as phyllotaxis, was studied as far back as ancient Greece, around 300 BC, and it was of interest to Johannes Kepler and Leonardo da Vinci as well. And in the late 20th century, a formal mathematical model was given that showed that you would observe these Fibonacci ratios as you're maximizing your exposure to sunlight.

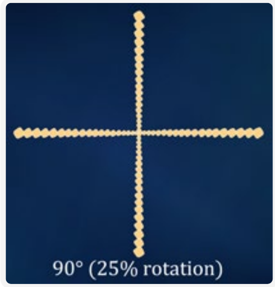
A similar phenomenon occurs when you count the number of spirals in pinecones and pineapples. For instance, in pinecones, it's often the case that either the number of clockwise spirals or counterclockwise spirals will be 13, and the number of spirals in the other direction will either be 8 or 21.

There are some natural objects, including some pinecones, that will produce spiral numbers that are double the Fibonacci numbers. And there are some objects that are counted by the Lucas numbers. But in all these cases, they still use the same Fibonacci growth rule: The ratio of consecutive terms gets closer and closer to ϕ.

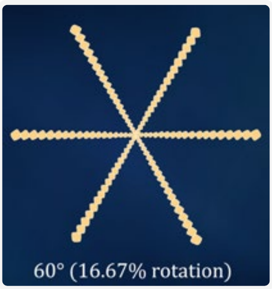
Consider the following simple method for creating sunflower patterns one seed at a time. You start with a seed in the center of the sunflower. Let's call this seed 1. Next, seed 2 emerges from underneath seed 1 and pushes it out of the way—let's say due north. Next, seed 3 emerges and pushes seed 2 out of the way. If seed 3 and all future seeds pushed in the same northward direction, then the seeds would produce a boring and not-very-efficient line.

To avoid this, each new seed rotates by a fixed angle and pushes the middle seed in that direction. For example, if you start with seeds 1 and 2 as before, where seed 2 pushes seed 1 in the due-north direction, if you use a clockwise angle of 90° (¼ of a circle, or a 25% rotation), then seed 3 will push seed 2 in the due-east direction. Then, seed 4 will push seed 3 in the due-south direction. Seed 5 will push seed 4 in the due-west direction, and seed 6 will push seed 5 in the due-north direction, and the process continues.

If you repeat this process, with a 90° (or 25%) rotation, then the sunflower would look something like this, which is not very efficient either.


90° (25% rotation)

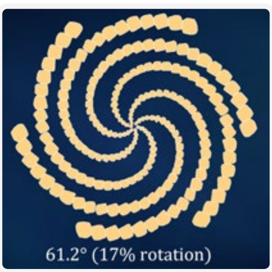
Notice that the seeds on the outside are a little larger than the seeds near the middle since they are older and have had more time to grow.


60° (16.67% rotation)

Suppose you rotate by 60° instead; then you'd get an arrangement that looks like this, which makes sense because 60° is $^1/_6$ of a circle. To avoid getting straight lines, you should avoid fractions of the circle that have a small denominator.


122.4° (34% rotation)

Now try an angle that's 34% of the circle, or 122.4°. This time you get something much more curvy emanating from 3 spirals since the number 0.34 is so close to the fraction $^1/_3$.


61.2° (17% rotation)

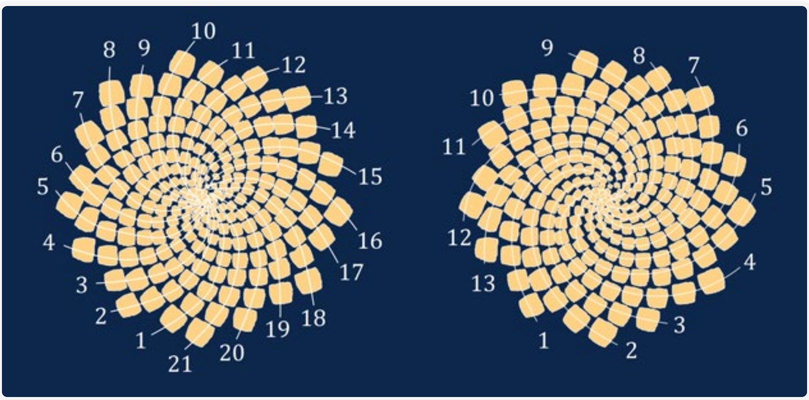
If you cut that angle in half and rotate 17% of the circle, or 61.2°, you get an even more efficient use of space, arising from 6 spirals. But there's still a lot of unused space. What angle should you use to pack in the most seeds?

You can generate these images yourself at www.MathIsFun.com. (Search for *Fibonacci* on the website.)


222.5° (61.8% rotation)

Let's try the golden angle of 61.8% of the circle; that's $^{360}/_{\phi}$, or 222.5°. Now you get a seed packing that looks like this. You get a similar pattern with the other golden angle of 137.5°, but it's spiraling in the opposite direction.

If you count the number of spirals in the clockwise direction, you get 21 spirals. And when you count in the counterclockwise direction, you get 13 spirals, so as predicted, you get consecutive Fibonacci numbers!

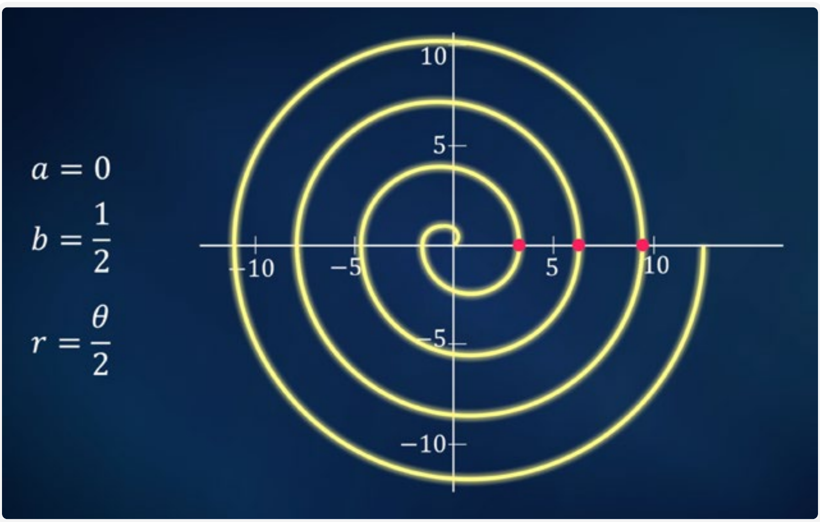


How do the sunflowers know to do this? Since the golden angles produce the most seeds in a fixed region, then it makes sense that it would evolve this way through the natural selection process.

Archimedean, Golden, and Logarithmic Spirals

Spirals have fascinated mathematicians since the ancient Greeks studied them more than 2000 years ago. A spiral is a curve on a flat plane. It winds around a point with a distance that gets larger when the angle increases. The simplest spirals are called Archimedean spirals. These spirals can be described by an equation of the form $r = a + b\theta$, where a and b are positive numbers; r is the radius, which denotes the distance from the center; and θ denotes the angle.

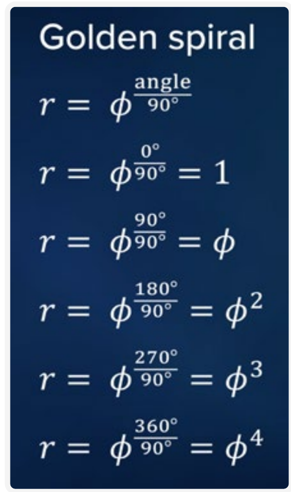
For example, when $a = 0$ and $b = ½$, you get the Archimedean spiral $r = \theta/2$. These spirals look like a cinnamon roll or a rolled-up carpet. Notice that at each full turn, the radius grows by a constant amount. In this example, with each full turn, the distance from the center grows by exactly π. So, after 1 full turn, the radius is π; after 2 turns, it's 2π; after 3 turns, it's 3π; and so on.

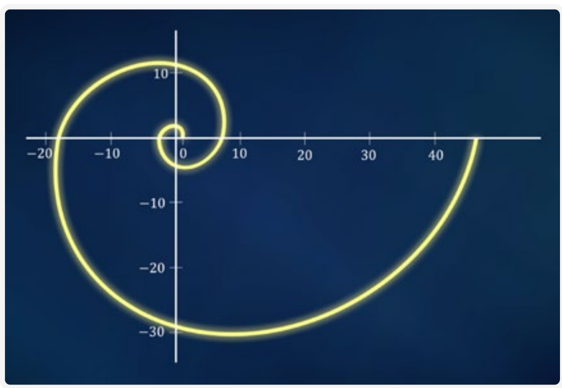


In this course, you've learned about the golden rectangle, along with golden triangles, gnomons, angles, and more. So, naturally, there must be a golden spiral.

The golden spiral has formula $r = \phi^{\text{angle}/90°}$. Notice that at every ¼ turn, the distance grows by a multiplicative factor of ϕ, and after each full turn, it grows by a factor of ϕ^4.

Here's what the golden spiral looks like.





Notice that the golden spiral is not an Archimedean spiral. This graph looks more like the shell of a mollusk than a rolled-up carpet. In an Archimedean spiral, at each full turn, you add a fixed amount to the radius. In the golden spiral, after each full turn, you multiply the radius by a fixed amount. The golden spiral is an example of a logarithmic spiral, which is of the form $r = ae^{b\theta}$, where e is the exponential number 2.71828… and the angle θ is measured in radians. With some algebra, it can be shown that the golden spiral can be put in this form, where $a = 1$ and $b \approx 0.306$.

Logarithmic spirals show up in nature a lot because of their many special properties, and they're sometimes referred to as growth spirals or equiangular spirals. One of their properties is self-similarity, which means that as the spiral moves in or out, it retains the same basic shape. As you zoom in continuously on a logarithmic spiral, you would see the same shape repeating itself.

Spirals arise naturally in biology and astronomy. For instance, the nautilus shell is a logarithmic spiral, as are the arms of spiral galaxies and cyclones. Golden ratio worshipers use examples like these to claim that the secrets of the universe are somehow revealed through the mystic number ϕ. If you rearrange the letters of *golden ratio*, you get *god relation*. Unfortunately, although nautilus shells and galaxies often do have spirals, they are rarely of the "golden" variety.

Just like there are lots of rectangles in existence, only a fraction of them are golden rectangles. The same is true with logarithmic spirals. This phenomenon is seen elsewhere in nature. The arms of spiral galaxies and cyclones can be logarithmic spirals (but there are other kinds of spirals, too, where the angle changes as you rotate around it), and even when they are logarithmic spirals, they are rarely the golden spiral.

These same mystics will also claim that the golden ratio shows up in the natural world far more often than it actually does, claiming that the human body is a living testament to the number ϕ. For example, it's been said that a person's height from head to toe divided by the height of their belly button is equal to the golden ratio, or the ratio of the length of shoulder to fingertips versus elbow to fingertips is always the golden ratio. Such claims, when they've been examined, have been found to be nothing more than wishful thinking or very generous rounding of numbers.

Many golden ratio myths were popularized by the author Dan Brown in his bestselling book *The Da Vinci Code*. It's true that Leonardo da Vinci did create the illustrations for Luca Pacioli's book on the divine proportion, but it's debatable whether he deliberately incorporated these images in his work. Nevertheless, people continue to draw golden rectangles and spirals on many of his and famous artists' paintings and draw golden conclusions in this way.

Check out mathematician George Markowsky's article "Misconceptions about the Golden Ratio," in which he discusses several popular but erroneous beliefs about the golden ratio in architecture, art, and even human anatomy.

Art and Architecture

Another place where the golden ratio appears is in the works of art, design, and advertising—often accidentally, but sometimes intentionally due to the belief that the golden rectangle is somehow the rectangle with the most aesthetically pleasing proportions. This was motivated by the work of a 19th-century psychologist who attempted to prove this about golden rectangles and golden ellipses. But unfortunately, when modern psychologists attempted to replicate the experiment, they did not obtain the golden results.

But partly due to the belief that this rectangle was somehow inherently appealing, this has led artists, photographers, and advertisers to deliberately incorporate golden rectangles in their work. For instance, photographers often recommend putting the subject of your photo somewhere other than the direct middle of the photograph and advocate the rule of thirds, where you imagine breaking your photo into a 3-by-3 grid consisting of 9 squares—like a tic-tac-toe board—and putting the subject of your photo at one of the points of intersection.

Some photographers recommend breaking up the image in a golden way, where the grid lines are not equally spaced. So, instead of grid lines being cut into equal proportions, such as 1 to 1, instead they're cut into golden proportions, such as 1 to 0.618 to 1. This is called the ϕ grid.

Some photographers and artists recommend breaking up the image in a golden way by imagining a golden spiral that starts from one corner of the photo or painting and converging to a single important point while visiting visually interesting points along the way.

Another place where the golden ratio supposedly appears is in great works of art and architecture. Despite claims made by uncountably many websites, there is no evidence that the ancient Egyptians were aware of the golden ratio, much less celebrated that number in their pyramids—nor was it used in the construction of the Taj Mahal. And although the ancient Greeks were aware of the mathematical properties of the golden ratio, they made no claims that it symbolized beauty or ideal rectangles, and they did not celebrate that number in their architecture. And although it's unlikely that Leonardo da Vinci intentionally used the golden ratio or golden rectangles in his work, some artists, such as Salvador Dalí, did make explicit use of them.

There seems to be no evidence that the golden ratio was deliberately used by Mozart, Bach, Beethoven, or others—but it was possibly used by Claude Debussy. And although students of design and architecture are exposed to the concept of the golden ratio, it does not seem to be explicitly used very much. A major exception was the Swiss French architect who went by the name Le Corbusier, who was considered a pioneer of modern architecture.

There have certainly been many works of art and architecture that celebrate the Fibonacci numbers and the golden ratio. Fibonacci numbers have adorned buildings, railway stations, and even chimneys all over the world. There's just something about being surrounded by Fibonacci numbers that makes people smile!

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