THE GREAT COURSES®

Topic
Science & Mathematics

Subtopic
Mathematics

The Shape of Nature

Course Guidebook

Professor Satyan L. Devadoss
Williams College

PUBLISHED BY:

THE GREAT COURSES
Corporate Headquarters
4840 Westfields Boulevard, Suite 500
Chantilly, Virginia 20151-2299
Phone: 1-800-832-2412
Fax: 703-378-3819
www.thegreatcourses.com

Printed in the United States of America

Satyan L. Devadoss, Ph.D.

Associate Professor of Mathematics,
Williams College

Professor Satyan L. Devadoss was born in Madurai, India, on November 26, 1973, and moved to the United States in 1981. He graduated from North Central College in 3 years (in 1993, at the age of 19), during which he won the Outstanding Student of Mathematics award each year and was class valedictorian. He then went to graduate school at Johns Hopkins University, where he was the first recipient of the William Kelso Morrill Award (1995) for excellence in teaching mathematics. In 1997, he was awarded the Dean's Teaching Fellowship to design and offer a course of his choosing.

After receiving his Ph.D. in Mathematics in 1999, Professor Devadoss became a Ross Assistant Professor at The Ohio State University, where he created a course about shapes in nature and received the 2001 Freshman Research Seminar Award. In 2002, he was supported by the National Science Foundation (NSF) to attend the International Congress of Mathematicians in Beijing.

In 2002, Professor Devadoss joined the faculty in the Department of Mathematics and Statistics at Williams College. Here, he received NSF grants for his work in topology and computational geometry. In 2008, he was awarded North Central College's Young Alumni Award for excellence in his career and for demonstrating service to the community and the college.

Over his career, Professor Devadoss has designed more than a dozen novel courses in mathematics, computer science, and studio arts. He has organized art-math symposiums and undergraduate conferences and has been invited to give more than 50 lectures. He has supervised 8 undergraduate and graduate student theses and directed 4 NSF Research Experiences for Undergraduates programs at Williams College.

Professor Devadoss has published more than 12 papers, including work on configuration spaces, cartography, polytopes, origami, triangulations, and juggling. He has been invited to be a member of Mathematical Sciences Research Institute (MSRI), a think tank in Berkeley, California. He has also recently cowritten a book on computational geometry with Joe O'Rourke, one of the founders of the field.

Professor Devadoss is happily married (at least from his end), with 3 kids. He lives in Williamstown, Massachusetts. He cherishes his lack of exercise and the joy of eating ice cream. ■

Table of Contents

INTRODUCTION

Professor Biography i
Course Scope 1

LECTURE GUIDES

LECTURE 1
Understanding Nature 3

LECTURE 2
The Language of Shapes 6

LECTURE 3
Knots and Strings 9

LECTURE 4
Creating New Knots from Old 11

LECTURE 5
DNA Entanglement 13

LECTURE 6
The Jones Revolution 15

LECTURE 7
Symmetries of Molecules 18

LECTURE 8
The Messy Business of Tangles and Mutations 21

LECTURE 9
Braids and the Language of Groups 24

LECTURE 10
Platonic Solids and Euler's Masterpiece 27

Table of Contents

LECTURE 11
Surfaces and a New Notion of Equivalence 30

LECTURE 12
Reaching Boundaries and Losing Orientations 33

LECTURE 13
Knots and Surfaces 35

LECTURE 14
Wind Flows and Currents 37

LECTURE 15
Curvature and Gauss's Geometric Gem 40

LECTURE 16
Playing with Scissors and Polygons 43

LECTURE 17
Bending Chains and Folding Origami 46

LECTURE 18
Cauchy's Rigidity and Connelly's Flexibility.....................................49

LECTURE 19
Mountain Terrains and Surface Reconstruction................................52

LECTURE 20
Voronoi's Regions of Influence 55

LECTURE 21
Convex Hulls and Computational Complexity 57

LECTURE 22
Patterns and Colors...60

LECTURE 23
Orange Stackings and Bubble Partitions..63

Table of Contents

LECTURE 24
The Topology of the Universe 66

LECTURE 25
Tetrahedra and Mathematical Surgery 69

LECTURE 26
The Fundamental Group 72

LECTURE 27
Poincaré's Question and Perelman's Answer 75

LECTURE 28
The Geometry of the Universe 78

LECTURE 29
Visualizing in Higher Dimensions 81

LECTURE 30
Polyhedra in Higher Dimensions 83

LECTURE 31
Particle Motions 86

LECTURE 32
Particle Collisions 88

LECTURE 33
Evolutionary Trees 90

LECTURE 34
Chaos and Fractals 92

LECTURE 35
Reclaiming Leonardo da Vinci 95

LECTURE 36
Pushing the Forefront 98

Table of Contents

SUPPLEMENTAL MATERIAL

Glossary 100
Biographical Notes 102
Bibliography 105

The Shape of Nature

Scope:

The world around us is filled with intricate and amazing shapes, both seen and unseen. This course will not only provide a masterful guide to the sweep of designs in nature but will also equip us with concrete mathematical tools to tackle cutting-edge problems. We will focus on things of very small scale, learning about string theory and quantum entanglement in physics, about origami folding and how it shows up in protein design, and about knotting of DNA strands and how to untangle them. We will also look at nature from a grand scale, learning about general relativity and the curvature of planets and stars in spacetime and even about the shape of our universe and how we can try to find it. Using vivid visual imagery and motivated by real-world problems, we will explore mathematics with such simple ideas as tying strings and folding sheets of paper to bring us to the forefront of scientific research.

The vision for the course can be seen as a two-stranded braid. One braid focuses on shapes and designs that appear in nature and looks into the fields of biology, chemistry, physics, and more. The other braid provides the mathematical tools and ideas to understand, manipulate, and explore these shapes and designs. Although our motivation will always be linked to the natural world, we will sometimes take mathematical side roads to peek behind the machines that make things work. For example, the simple notions of addition and multiplication will be seen in a fresh perspective as they appear in the world of shapes. The overall structure of the course follows a mathematical perspective, building on simple structures and moving toward deeper complexity. Throughout these lectures, we present numerous unsolved problems in the world of shapes that are easy to state but thus far have resisted the attacks of talented researchers.

The introductory lectures set the stage for the main attractions. We look at the language of shapes as seen through the eyes of geometry and topology, then study the notion of dimension, where we see not only how to break shapes into levels of complexity but how they can interact with their environment. The course is divided into dimensional settings, starting with one-dimensional knots and links, moving to the geometry of two-dimensional surfaces, and exploring the world of three-dimensional manifolds. The closing lectures go beyond the standard notions of shapes and peer into the unseeable secrets of nature by considering *n*-dimensional objects, such as spaces of phylogenetic evolutionary trees, particle collisions, and string theory. From higher dimensions, we move to fractal dimensions and chaos, a concept undreamed of until a few decades ago. In the penultimate lecture, we explore the intersection of mathematics and the visual arts, looking at a number of great artists and following their works as they influence and are influenced by the scientific world around them. Finally, we take a look back at some of the ideas and results we have seen throughout the course and highlight some mathematical challenges for the 21st century. ■

Understanding Nature

Lecture 1

The most important lesson that we can learn in these entire series of lectures is that shapes determine purpose. Form and function are interrelated.

In this course, we will embark on an amazing journey in the study of shapes, ranging from big, puffy clouds to the patterns on ties and shirts. Why do we care about shapes? Because shapes determine purpose; in other words, form and function are interrelated. We will also bring in the power of a new field of mathematics to understand shapes.

Our focus will be on shapes in nature. It is crucial for both preserving and living in nature to gain a greater understanding of how nature works, and studying the shapes of nature will give us some insight into this topic. The study of nature has also opened numerous doors for civilization, providing the models for advances in science and technology.

Objects can be seen on several different levels. Those on our level in size are the most obvious, such as trees, pinecones, mountains, and insects. But nature also appears at the micro level, for example, in the structure of DNA, and at the macro level, with planets, black holes, and even the shape of the universe itself.

What does all this have to do with mathematics? Some complicated mathematical equations are related to capturing data about pictures, such as mathematical curves drawn on a plane, and for most of us, that's where the relationship between mathematics and shapes ends. But the power of mathematics is that it provides us with a language in which we can understand shapes far, far beyond numbers. As we proceed through these lectures, we will learn new tools to help us understand and manipulate shapes, just as we do with numbers and equations.

The goal of this course is to weave together ideas from nature and mathematics, with special focus on shapes that appear in biology, chemistry, and physics. The course is divided into 4 parts based on the complexity of the shapes. We will first look at **knots**, simple shapes based on lines and circles, which appear in DNA structures, string theory, knotted molecules, and mutations. As we increase complexity, we'll talk about **surfaces**, which appear in space telescopes, stent designs of arteries, and curvatures of mountain ranges. Next, we will move to higher-dimensional objects called **manifolds**. Examples of ideas related to manifolds include soap bubbles and Einstein's theory of relativity. Finally, we will look at superstructures. Here, we will struggle with ideas related to particle motions and collisions, the space of **phylogenetic trees**, and fractals and chaotic systems.

… the power of mathematics is that it provides us with a language in which we can understand shapes far, far beyond numbers.

This course only begins to explore the math underlying these concepts. The first basic mathematical idea we'll look at in understanding the language of shapes is the notion of equivalence. Can we say that 2 shapes, such as a sphere and a cube, are equivalent? In one sense, they are. They both have a smooth, connected surface, but the cube has sharp corners. What about a sphere and a donut? What about a small sphere and a large one? As we'll see in upcoming lectures, equivalence depends on the scale in which we're working and what we care about in our problem. ■

Important Terms

knot: A circle placed in 3 dimensions without self-intersections.

manifold: A generalization of a surface to higher dimensions, where each point on the manifold has a neighborhood having the same dimension.

phylogenetic tree: A mathematical tree structure that shows the relationship between species believed to have a common ancestor.

surface: An object on which every point has a neighborhood that has 2 degrees of freedom.

Questions to Consider

1. In what ways do you see mathematics used in the study of nature, either now or in the past?

2. What shapes are most fascinating to you? Make a list. Do certain common features appear in your list?

The Language of Shapes

Lecture 2

Topology is a part of geometry that focuses on the qualitative—not quantity but quality—in other words, the kind of position and the kind of underlying structure that defines the object.

We can think of mathematics as broken into 3 classes: analysis, algebra, and geometry. Analysis is the area of mathematics that studies change, for example, the growth of a given population. Algebra is used to study structure, and geometry is the study of shape. Of course, mathematics is far more complicated than this breakdown would imply. There are also combinations and specializations of these fields.

Topology is a relatively new branch of geometry that focuses on the qualitative, in other words, the position and underlying structure that define an object. In particular, the word to associate with topology is "relationships." The origins of topology can be traced back to the work of **Leonard Euler** and the problem known as the Seven Bridges of Königsberg. Euler's work of genius was to convert the map of the city of Königsberg into a graph. He showed that it was impossible to connect every region of the city with an even number of edges—one going in and one coming out. In solving the problem, Euler ignored concepts of area, length, volume, and angle, but he talked about relative position, how things were connected.

To create a finer distinction between geometry and topology, consider equivalences of shapes. Equivalence in geometry is based on rigid motions. For example, if we rotate an object, it is still the same object as it was originally. Equivalence in topology is based on a world called isotopy, sometimes called rubber sheet geometry. Here, we're allowed to stretch and pull an object, but we're not allowed to cut and paste. If we take an object and stretch it like clay, relative position does not change. Thus, from a topological point of view, a cube and a sphere are equivalent.

Different situations call for study through geometry or topology. Macro and micro problems are usually too hard for geometry but are possible for topology. We cannot answer questions about size, volume, and area in a macro or micro setting, but we can grasp relative position.

In closing, we look at the concept of **dimension**, which is simply a number that we associate to a shape. Dimension is broadly defined as degree of freedom—the smaller the dimension, the less freedom you have, and the greater the dimension, the more freedom you have. Dimension is narrowly defined as the amount of information needed to pinpoint a location. An example of a zero-dimensional world is a point; a line and a circle are both 1-dimensional, the plane is 2-dimensional, and our universe is 3-dimensional.

The important point to remember here is that dimension is simply a mathematical construct; it associates a number to a shape. Thus, dimension doesn't have to stop at 3. Maybe our imaginations stop at understanding things beyond 3 dimensions, but dimension does not need to stop there. In the next lecture, we'll begin a further exploration of dimension with 1-dimensional objects called knots. ■

Equivalence in topology is based on a world called isotopy. This is sometimes called rubber sheet geometry. Here, we are allowed to stretch and pull on our object, but we're not allowed to cut and glue.

Name to Know

Euler, Leonhard (1707–1783): One of the greatest mathematicians of all time, his scientific works cover analysis, number theory, geometry, and physics. He was one of the first to use topology, from which we receive the formula $v - e + f = 2$ of a polyhedron.

Important Term

dimension: An invariant given to a point on a shape that measures the degrees of freedom afforded at that point.

Questions to Consider

1. Which feature of an object, its geometry or its topology, do you seem to notice instinctively? Why?

2. What is the dimension you are most comfortable dealing with? Is it three dimensions, given that you are a three-dimensional being? Or is it one or two dimensions?

Knots and Strings

Lecture 3

We've proved a truly beautiful result based on shapes. It's based on Reidemeister moves, which all are local phenomena that control a global structure. It is based on colorings—simple ideas that we had when we were kids, applied in a powerful way. This is the type of creativity and originality that pushes the frontiers of math.

In this lecture, we consider the simplest and most elegant of shapes, 1-dimensional ideas of circles. For a topologist, a circle is simply of piece of string in which the ends are connected. Our goal is to study how such circles sit in 3-dimensional space.

The most beautiful examples of circles in 3 dimensions are knots, which are defined as circles that are placed in 3 dimensions in different ways. The most elegant way to construct knots is simply to close up the ends of strings. The simplest form of knot is a circle, called the unknot. Another knot is the trefoil. By isotope—by just stretching and pulling without cutting—we cannot make a trefoil into an unknot. Thus, these 2 objects seem to be very different topologically.

The best way to study knots is to look at their projections on paper. The tools needed to study projections were provided by Kurt Reidemeister, who developed 3 moves that change the knot projection but not the knot it represents. According to Reidemeister move I, we can twist a vertical string one way or the other and still have the same knot but with different projections. With Reidemeister move II, we can push one vertical line behind another, introducing 2 new crossings and, again, changing the knot projection but not the underlying knot. With Reidemeister move III, we can move any strand that's behind or in front of a crossing below that crossing. In other words, a crossing at the strand above or below is independent of where it's placed. These 3 moves are the only ones needed to go from one projection of a knot to any other projection of the same knot, regardless of how complicated we make the sequence of the 3 moves. Reidemeister moves can help us tell whether 2 knots are the same, but they do not help us tell knots apart.

A major breakthrough using Reidemeister moves was introduced by Ralph Fox in the 1950s with the idea of coloring. According to Fox, a projection of a knot is 3-colorable if it meets 3 conditions: If we can color every strand (a piece of knot that goes from one crossing to another) using one of 3 colors, if all 3 colors are used in the knot projection, and if either all 3 colors or only one color meet at each crossing. We see some projections in which the unknot is not 3-colorable, but the trefoil is.

The most beautiful examples of circles in 3 dimensions are what we call knots. Knots are simply defined to be circles that are placed in 3 dimensions in different ways. The most elegant way of constructing knots [is] by taking strings and just closing up the ends of the string.

How do colorability and Reidemeister moves fit together? According to Reidemeister, if one projection of the knot is 3-colorable, then every time we do a Reidemeister move, it stays 3-colorable. Further, every possible projection of this knot will be 3-colorable. In other words, 3-colorability is not a property of the projection of the knot but a property of the knot itself. The idea of 3-colorability allows us to see that, in fact, the unknot and the trefoil are different knots. ■

Suggested Reading

Adams, *The Knot Book*.

Questions to Consider

1. How many different knots can you draw on paper where the knot crosses itself five times or fewer? What about crossing six times?

2. Can you generalize the coloring method to use four colors rather than just three?

Creating New Knots from Old

Lecture 4

The fundamental problem in knot theory is to find stronger and stronger invariance. We want to tell apart more and more knots somehow, like a taste test.

This lecture begins by manipulating knots with the idea of addition. We can add 2 knots by making 2 cuts on the boundary of knot 1 and 2 on the boundary of knot 2, then gluing those cuts together to form a new knot. Knot addition is a small operation because we cut only on the boundary, and we introduce no new crossings. The new knot we have created is called a composite knot. We can also identify an "unknot," a concept similar to 0, that allows us to add a knot to it and get the same knot. We cannot, however, remove the complexity of knots using addition, as we can remove complexity with numbers. In other words, there is no subtraction of knots.

The notion of prime numbers can also be extended to knots. A composite knot is the sum of 2 nontrivial knots. A prime knot is not a composite knot; in other words, it can't be broken into 2 distinct pieces.

One goal in trying to tell knots apart is to find invariance. A knot invariant is a property, such as tricolorability, that is assigned to a knot and does not change as the knot is deformed. A knot invariant can distinguish between knots that are different but not those that are equivalent. The fundamental problem in knot theory is to find stronger invariants—to be able to distinguish among more knots. The dream is to find a characteristic for each distinct knot, which would be called a complete knot invariant.

The crossing number and the unknotting number are 2 classical knot invariants. The crossing number $c(K)$ assigns the least number of crossings that appear in any projection of the knot and is very difficult to find. A famous unsolved problem is to show whether the crossing numbers of 2 separate knots are related to the crossing number of the 2 knots added together.

The second classic knot invariant is the unknotting number $u(K)$, which is the least number of crossing changes in any projection needed to make the knot into the unknot. Just like the crossing number, the unknotting number is difficult to calculate, and it also has a famous unsolved problem: Is the unknotting number of one knot plus the unknotting number of a second knot equal to the unknotting number of the 2 knots put together?

This leap-frog relationship [between math and physics] is a fantastic way of research progressing, one motivating and pushing the other, like a big brother encouraging the younger brother, taking turns as to who's better.

Do the unknotting number and the crossing number both measure the same kind of complexity? Our intuition says that if we have a projection with the least number of crossings to draw, it's most likely the projection with the least number of strands to uncross, but that's not the case. Both numbers measure complexity, but they measure different kinds of complexity. When we're talking about the simplest projection of a knot, we have to know what we mean by simplest—is it the simplest to draw or the simplest to untangle? ■

Suggested Reading

Adams, *The Knot Book*.

Questions to Consider

1. In adding numbers, 3 + 4 = 4 + 3. Is this true for knots?

2. Can the unknotting number of a knot be greater than its crossing number? Why or why not?

DNA Entanglement

Lecture 5

The structure of the DNA is extremely elegant, formed by the shape of this double helix with 2 strands perfectly interweaving between each other.

So far, we've looked at one circle knotted, but this lecture considers links, which are several circles knotted together. They can be individually knotted, or they can be tangled together in different complexities. We see, for example, the unlink (analogous to the unknot), the Hoff link (the link that forms chains), and the Whitehead link.

Similar to knots, links are equivalent up to isotopy. We can take the links in our hands and stretch them, but we're not allowed to pull or cut. For their projections, we again have Reidemeister moves. We can also identify links with more than 2 components, such as the Borromean rings. As it was with knots, one of our goals with links is to find link invariants—things based on the link itself that don't change. The number of components in a link is the first linking invariant. For example, the simple chain link has 3 components. The unlink, Hoff link, and Whitehead link have 2 components.

The linking number is a second invariant. To compute the linking number, we must first orient each component of the link by choosing a direction to travel. Next, we look at crossings between different components of the link. At every crossing, where 2 separate components meet, we give either a +1 value or a –1 value based on how we oriented the knot. We then add all the values at every crossing and divide by 2. Note that if we choose the orientations arbitrarily, the linking number values change. Thus, we must take the absolute value of our calculation. Note, too, that the Reidemeister moves do not change the linking number values.

We know from Reidemeister's theorem that we can get from any projection to any other projection with these 3 moves. That means that if we compute the linking number for this projection, we have computed the linking number for every projection possible and have found the linking number of the link itself. The linking number is an invariant.

How can we apply the linking number—this +/–1 crossing information—to knots? We call this calculation the writhe of a knot. We first orient the knot, then obtain a value at every crossing, and add the values. We don't need to divide by 2 because we don't have another component to worry about. And we don't need to take the absolute value because if we change the orientation, the writhe doesn't change. Reidemeister moves II and III don't change the writhe, but Reidemeister move I does, which means that the writhe is not a knot invariant. The writhe measures the kind of twist a belt goes through when a Reidemeister move I is performed on it. Thus, the writhe is not invariant for knots, but it is invariant for ribbons. Indeed, DNA can be thought of as a twisted ribbon in this helix.

This idea of adding up crossing information sounds beautiful, and it feels like we can use this beautiful idea on other things. Remember, we were motivated by this from the DNA linking. We naturally ask, why not try this idea on knots?

We have come full circle. We were motivated by looking at DNA with its double strand, which led to links. Then we then came up with the linking number, which was an invariant, and we pushed it on to knots to find the writhe of these ribbon strands. ■

Suggested Reading

Adams, *The Knot Book*.

Questions to Consider

1. Can you generalize the Borromean rings for four components so that if you remove any one component, all the other pieces become unlinked?

2. Is it possible to compute the linking number of a knot rather than a link?

The Jones Revolution

Lecture 6

How powerful is this polynomial? A big, open question has been … can we tell any knot from the unknot? Any time we use the Jones polynomial, we get the value of the unknot as 1. But any time we use the Jones polynomial of any other knot … the Jones polynomial is not 1. It seems that the Jones polynomial is able to tell every knot apart from the unknot.

In this lecture, we will turn to the power of algebra, which measures structures in the world of topology, and study a new algebraic knot invariant. This invariant does not assign a number or a property to a knot. Instead, it assigns a polynomial. This polynomial is called the Jones polynomial after its discoverer, **Vaughan Jones**, who found it in 1984.

A polynomial, such as $5a^3 + 4a + 2$, can actually be thought of as simply a set of numbers. In this case, the set is 5,0,4,2; 5 for the amount of a^3, 0 for the amount of a^2, 4 for the amount of a, and 2 for the amount of the constant that is not a. With this in mind, we will build the Jones polynomial (the bracket polynomial), using a method developed by Lou Kauffman.

The bracket polynomial is based on 3 rules. The first rule tells us that the bracket value of the circle, which in the projection we see is the unknot with no crossings, equals 1. The second rule tells us the relationship between the 3 polynomial values. Whenever we get one polynomial, we can multiply it by a of the other polynomial plus b times the third polynomial, and the equality will work. We can also think about the second rule as follows: If we have a positive slope, we cut vertically and horizontally; if we have a negative slope, we cut horizontally first, then vertically. The third rule gives us another relationship, that between a link with an extra circle and the polynomial of just the link.

If we're given a knot or a link and we use the second rule repeatedly, it keeps removing all our crossings and makes vertical and horizontal cuts. If we keep applying the rule, we are left with a collection of circles because all the crossings are gone. Then, the third rule says that every time we have a collection with a free-floating circle, we can throw it away as long as we have multiplication by c to the polynomial. The second rule destroys all the crossings into circles; the third rule gets rid of the circles; and the first rule says that if we're left with one circle at the end, its value is 1.

Edward Witten, … a superstar physicist, who himself won a Fields Medal, related the work of the Jones polynomial that Vaughan Jones came up with to work in string theory and in 3-dimensional objects.

Our goal is for these rules to satisfy some kind of knot invariant properties. We want to make sure that the polynomial does not change as we perform Reidemeister moves. With a great deal of algebraic manipulation, we see that the polynomial is invariant for Reidemeister move II. Moreover, Reidemeister move III emerges easily out of the work we performed for move II. Unfortunately, Reidemeister move I fails us again, just as it did when we were working with the writhe. However, we can combine both the negative properties that cause Reidemeister move I to fail and make the invariance work.

Any time we use the Jones polynomial, we get the value of the unknot as 1. Any time we use the Jones polynomial of any knot that is not the unknot, the value is not 1. Although it's still an open question, it seems that the powerful Jones polynomial is able to tell every knot apart from the unknot. ■

Names to Know

Jones, Vaughan (1952–): Winner of the Fields Medal in 1990, he created one of the most powerful knot invariants.

Witten, Edward (1951–): A mathematical powerhouse who received the Fields Medal in 1990, Witten is considered the greatest physicist of our time, known for his work in string theory.

Suggested Reading

Adams, *The Knot Book*.

Questions to Consider

1. Compute the Jones polynomial of a knot of your choice.

2. How would things change if we defined the bracket <0> of the circle to be a value other than 1?

Symmetries of Molecules

Lecture 7

A knot or a link is called amphicheiral if it can be made or deformed into its mirror image. What we're interested in is examining the mirror images of knots since they might give different chemical properties for these topological stereoisomers.

A broad topic in the study of shapes is the idea of symmetry and the question of whether the mirror images of 2 objects are equivalent. This lecture shows how this question relates to work on molecular compounds and topological stereoisomers in chemistry.

We begin with a review of calculations of the *X*-polynomial. For example, we compute the bracket polynomial of a double twist, the unknot, the Hoff link, and the trefoil, as well as the *X*-polynomial of the double twist and the trefoil. Note that any time we have a complicated knot, if we know how it works previously in a simpler version, we can use that value to compute the *X*-polynomial of the more complicated knot.

The *X*-polynomial has another stunning feature that relates to additions of knots. In a previous lecture, we asked the question: How are the crossing number and the unknotting number related to knot addition? The beautiful feature of the *X*-polynomial is this: The *X*-polynomial of knot 1 + *X*-polynomial of knot 2 = the *X*-polynomial of knot 1 × the *X*-polynomial of knot 2. Once we understand how prime knots work—the basic building blocks of knots—we can get polynomials for composite knots by this simple procedure. If we have a complicated composite knot, we just break it up into its prime pieces, compute each one separately, and multiply the answers together.

Before we continue, let's look at a pair of molecules, each with an identical number of atoms and identical atomic bonds but different placement in space. Such pairs are called topological stereoisomers. From a chemistry point of view, these 2 seemingly identical objects might have different properties. Chemists are interested in topological stereoisomers because they provide a means to create entirely new substances. In the study of knots, a knot or a link is called **amphicheiral** if it can be made or deformed into its mirror image. The mirror images of knots might give different chemical properties for these topological stereoisomers.

We see that Reidemeister moves can yield a mirror image of the figure 8 knot but not the trefoil. Notice that a projection of a knot and its mirror image are identical. They look the same, except that every positive crossing is replaced by a negative crossing. Thus, we need a tool that can tell whether each crossing is negative or positive. It turns out that the *X*-polynomial is beautifully designed to help do that.

The way multiplication works of polynomials somehow captures exactly how knots are put together. … The way these knots are glued and the way this new knot is formed is exactly captured by polynomial multiplication. This is gorgeous! It is a stunning and elegant result.

For any knot and its mirror image to be amphicheiral, the *X*-polynomial must be a palindrome. For example, the *X*-polynomial for the figure 8 knot is: $a^8 + a^4 - 3 + a^{-4} + a^{-8}$. If we take a mirror image of this knot, then all the *a*'s become *a* inverses and all the *a* inverses become *a*'s; we get the same polynomial because the figure 8 and its mirror image are the same.

Earlier, we calculated the value of the *X*-polynomial of the trefoil to be $-a^{16} + a^{12} + a^4$. By switching *a* and *a* inverse, we get $-a^{-16} + a^{-12} + a^{-4}$, which is the *X*-polynomial for the mirror image of the trefoil. These 2 polynomials aren't the same, which means that the mirror images of the trefoils are fundamentally different. Only the *X*-polynomial gives us the power to find this result. ■

Important Term

amphicheiral: An object is amphicheiral if it can be made into its mirror image.

Suggested Reading

Adams, *The Knot Book*.

Questions to Consider

1. What properties or traits do we use to tell an object apart from its mirror image? Which objects in our daily lives are equivalent to their mirror images?

2. Show that X(trefoil + trefoil) = X(trefoil)X(trefoil)

The Messy Business of Tangles and Mutations

Lecture 8

For Conway, there are 3 main types of tangles for us to focus on. … There's the infinity tangle. … There's the 0 tangle, … and then based on these 2 tangles, Conway said you can come up with the *n* tangle, where *n* represents any integer that you want.

Your cells, as they're dividing and multiplying, need to copy DNA information located in one cell into another. But as we know, DNA is a tangled mess. How can this structure be copied without too many errors? The answer is that an enzyme called topoisomerase alters the topology of the DNA. It straightens the DNA at a small local area to make the copying easier, then cuts, twists, changes crossings, and so on.

Error is sometimes introduced in the copying process, and when it is, mutations occur. In this lecture, we will look at the mathematical version of mutations. The mathematical notion of tangles, developed by **John Horton Conway** in the late 1960s, is one means of understanding mutations. A tangle is defined as a part of a projection of a knot, or link, around the circle, crossing it exactly 4 times.

The Reidemeister moves can be used with tangles under one condition: Tangles can be moved around only inside the circle. We say that 2 tangles are equivalent if one can be made into the other by Reidemeister moves within the circle. We can connect any tangle and make it into a knot or a link by connecting together the 2 northern and the 2 southern strands.

Conway's notation for knots starts with the simplest tangles and builds to more complicated ones. For Conway, there are 3 main types of tangles to focus on: the infinity tangle, the 0 tangle, and the *n* tangle, where *n* represents any integer. For example, a 2 tangle is simply a tangle that has 2 twists in it. To distinguish between a $-n$ tangle and a $+n$ tangle, we look at the overcrossing. If it has a positive slope, the tangle has a positive value; if it has a negative slope, the tangle has a negative value. This is the building block that Conway uses to construct rational tangles. Such tangles can be

… if you're given 2 tangles with the Conway notation, if you can find the continued fraction values and they turn out to be the same, this means that the tangles—the pictures themselves—must be the same.

constructed using rotations and reflections or with a method based on the number of integers in the notation. However, the notation itself is not a complete tangle invariant.

Conway's theorem relating to the equivalence of rational tangles states: Two rational tangles are equivalent if and only if their continued fraction values (derived from the notation) are the same. Remarkably, the continued fraction value embodies the shape of the tangle itself. Moreover, knots or links formed from these tangles will also be identical.

Tangles lend themselves beautifully to the arithmetic operations of addition and multiplication. Any tangle formed by addition and multiplication is called an algebraic tangle. Just as numbers do, tangles have an additive identity, the 0 tangle, but the multiplicative identity and the associative property fail for tangles.

We can perform 3 possible mutations of a tangle in a knot. These mutations are a mathematical approach to capturing what happens in DNA structures. Unfortunately, mutation operations are extremely hard to distinguish. Greater tools are needed to crack this problem in both mathematics and nature. ■

Name to Know

Conway, John H. (1937–): Conway is a professor at Princeton and a prolific mathematician whose works encompass geometry, group theory, number theory, and algebra. In particular, he is known for his Conway notation for knots and links.

Suggested Reading

Adams, *The Knot Book*.

Questions to Consider

1. Construct some mutations and show that the Jones polynomial remains the same for both mutants.

2. What does mutation do to the Conway notation of a knot?

Braids and the Language of Groups

Lecture 9

[Groups are] one of the most important mathematical structures. They're this algebraic system that keeps track of numerous things. They appear in several areas, not just in mathematics, but in chemistry, in symmetries of molecules, and in theoretical physics and quantum mechanics.

So far, we have been focused on knots and links, but now we move on to the study of braids. Braids are one of the oldest forms of pattern making in the world and are vital to physics and cryptography. They also further our understanding of knots and links.

A braid is a set of *n* strands attached between 2 horizontal bars; *n* can be any positive integer. In a braid, the strands must always flow down from the top horizontal bar to the bottom horizontal bar. Each crossing in a braid is not just keeping track of switching data, but it's also keeping track of over and under information.

Braids are classified based on the number of strands. We can manipulate the braids only between the 2 bars, just as the strands of the tangle had to stay inside the tangle circle. Moreover, just as we joined the corners of the tangle to form links and knots, we can also extend the strands on top of the braid around back of the bar, closing the braid to form links and knots. The trefoil, for example, is a simple 2-stranded braid with 3 twists in it. In fact, every knot and link is a closure of some braid.

Just as we created a language for rational tangles, we can also create a language for braids. Focusing on a braid with 3 strands, we cross the first strand over the second strand; this move is called sigma 1. The opposite move is sigma 1 inverse. Switching the second and third strands is sigma 2, and the opposite move is sigma 2 inverse. We can create braids using these 4 "letters" as a braid alphabet.

The larger algebraic structure that we're talking about is called a **group**. A group is a set of elements and a way of combining elements. Think of the group as a bag, and we need some way of putting 2 elements in the bag together to get another element in the bag. We call this an operation that works on these elements in the group.

A group must satisfy 3 properties: It must have an identity element (*e*); the multiplication operation must be associative; and every element in the bag must have an inverse such that when the element and its inverse are put together, the result is the identity. The integers under addition form a group, but the integers under multiplication do not. Neither knots nor tangles form groups under addition or multiplication, but braids do. We check identity, associativity, and inverses in a braid with 4 strands to show that braids form groups.

Every knot and link that you can possibly come up with is a closure of some braid. … If you want to know how knots and links work, you can ask the same question to the braids, and if you have full understanding [of] how braids work, it turns out you'll have a really good understanding [of] how knots and links work.

We can measure the equivalence of braids using our language and 3 rules. According to the first rule, sigma 1, sigma 1 inverse = sigma 1 inverse, sigma 1. In other words, if we put 2 inverses next to each other, they cancel out. This is just like a Reidemeister II move. The second rule says: sigma I, sigma I + 1, sigma I = sigma I + 1, sigma I, sigma I + 1. This is like a Reidemeister III move. The third rule does not relate to a Reidemeister I move; instead, it says: sigma I, sigma J = sigma J, sigma I. We can switch any 2 sigmas as long as I – J is greater than or equal to 2. Just by using these 3 rules and working with letters, we can change one word of a braid into another word of a braid and show that the 2 are equivalent. ■

Important Term

group: An algebraic structure given to a collection of elements with a means of combining the elements (composition) satisfying 3 conditions (identity, inverse, associativity).

Suggested Reading

Adams, *The Knot Book*.

Questions to Consider

1. Draw your favorite knot and try to find two different braids whose closures both result in the same knot.

2. Does the set of real numbers under addition form a group? How about under multiplication?

Platonic Solids and Euler's Masterpiece

Lecture 10

The most famous of all polyhedra are the 5 platonic solids. … These are: the tetrahedron, … made up of these 4 triangles; the cube, made up of these 6 squares; we have the octahedron, made up of these 8 triangles; we have the dodecahedron, made up of 12 pentagons; and we have the icosahedrons, made up of 20 triangles.

In this lecture, we move from 1-dimensional objects, such as knots and tangles, to 2-dimensional objects called surfaces. All the objects around us are 3-dimensional, but we observe only their 2-dimensional surfaces. We'll begin by looking at the sphere.

We start with the representations of the sphere with which we are most familiar, polyhedra. Polyhedra are **isotopic** to spheres. The advantage of working with polyhedra is that they have a finite number of flat faces, they have corners (vertices), and their faces meet along edges. The most well-known polyhedra are the 5 platonic solids: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. Each of these satisfies 2 conditions: all their faces are identical and the same number of faces meets at every vertex. These solids all appear in nature, for example, in crystal formations.

If we try to build these objects by construction, we can prove that we get only 5 of them. For example, if we put 3 triangles together along the edges, we get a tetrahedron; 4 triangles, an octahedron; and 5 triangles, an icosahedron. If we try to put 6 triangles together at a corner, however, the object becomes flat.

The cube has 3 faces meeting at one corner. The opposite, or dual, of the cube is the octahedron. The octahedron has 4 triangular faces meeting at a corner. The 4 for the number of faces and the 3 for the triangle make a 4, 3 combination. For the square, the 3 for the number of faces and the 4 for the sides make a 3, 4 combination. These objects are dual to each other; they look different, but they somehow capture the same data.

An amazing pattern exists for polyhedra, as follows: the number of vertices + faces = the number of edges + 2. For example, the tetrahedron has 4 vertices, 4 faces, and 6 edges: 4 + 4 = 6 + 2. This topological formula is called Euler's formula in honor of its discoverer. It governs the ways these objects can partition, or cut apart, spheres. We can prove this formula operationally by breaking any polyhedron into triangles, removing one triangle, laying the shape out flat, and deleting outside triangles until we have only one left. During the entire process, the value $v - e + f$ never changed. At the end, we get $v - e + f = 2$, which means that it was equal to 2 during the entire process.

An amazing pattern exists for polyhedra. … If we count the number of vertices, edges, and faces of a platonic solid, we always get that the number of vertices + faces = the number of edges + 2.

This formula has applications in chemistry, notably to the molecules called fullerenes. A fullerene is basically a family of molecules formed entirely of carbon atoms, in which each atom has exactly 3 bonds coming to it and the faces must be pentagons or hexagons. In other words, the vertices are carbon atoms, the edges are bonds, and the faces we get after building these atomic bonds are made up of pentagons and hexagons. Remarkably, no matter how we build a fullerene—even using the most complicated structure of carbon atoms possible—as long as it satisfies the 2 conditions of being built out of carbon atoms with 3 bonds at every corner and using only pentagons and hexagons, the resulting shape must have exactly 12 pentagons. The golf ball is an example of this result: As long as a golf ball has divots made up of pentagons and hexagons, not matter how many divots we have, 12 of them exactly must be pentagons and the rest must be hexagons. ■

Important Term

isotopic: A notion of equivalence, the strongest in the world of topology. Two objects are isotopic if they differ by stretching (rubber sheet geometry).

Suggested Reading

Cromwell, *Polyhedra*.
Richeson, *Euler's Gem*.

Questions to Consider

1. Draw three random polyhedra and show that Euler's formula works for all three of them.

2. If we are allowed to have a polyhedron with only squares and hexagons, does Euler's formula provide a relationship between the number of squares and the number of hexagons?

Surfaces and a New Notion of Equivalence
Lecture 11

If we had a hard time telling knots apart based on isotopy, surfaces are going to be much harder. They're 2-dimensional objects. Thus, a new concept of equivalence is needed—[homeomorphism].

This lecture looks at 2 questions: What possible shapes could the Earth have been had it not been a sphere? How would we know the shape of the Earth if we were not allowed to leave it? We begin by asking another question: What do we mean when we say "surface"? A surface must look the same at every point. This is called the neighborhood condition. A surface must also be finite in area.

What do we mean by surfaces being equivalent? With knots, we talked about equivalence as being isotopic. We can stretch and pull, but we cannot cut and glue. If we apply the same definition to surfaces, we can see that the power of isotopy is such that it doesn't tell us much about equivalence of surfaces. We will need a new concept of equivalence to tell us whether 2 surfaces are the same or different. This new concept is called homeomorphism, meaning "similarity of form." Two surfaces are **homeomorphic** if we can cut surface 1 into pieces, pull the surface apart based on these cuts, manipulate each piece we choose up to isotopy, then glue the pieces exactly the way we cut them—along the same seams—and get surface 2.

Homeomorphism is a weaker notion of equivalence. In isotopy, we're not allowed to cut and glue. Here, we can cut and glue, but we have to reglue the same way we cut. Under homeomorphism, all knots are the same. We can cut a knot at any point, then stretch and pull to untangle the knot, reglue it, and get the unknot.

What does homeomorphism actually measure? What surfaces are equivalent under homeomorphism, and what is it trying to say? To understand this, we need to consider the difference between extrinsic and intrinsic. Extrinsic is based on what your world looks like if you're an outside viewer. Intrinsic is based on what your world looks like if you live on the surface. Isotopy measures your world the way you would look at it if you left the world. Homeomorphism measures your world the way you would look at it if you lived in the world. With the cutting and gluing, though it looks like you're shattering the world, at the end of the day, since you have to glue the same way you cut, what would once be apart would then be put back together identically. Isotopy is deformation in the extrinsic world, whereas homeomorphism is deformation in the intrinsic world.

Isotopy measures worlds the way you would look at it if you left the world. But homeomorphism measures worlds the way you would look at it if you lived in the world.

Given a surface, we learn that no matter how we partition the surface into pieces, the value we get from the number of vertices minus the number of edges plus the number of faces based on the partition (a value called the Euler characteristic) does not change. In other words, we can partition the surface any way we want into vertices, edges, and faces, and this $v - e + f$ does not change; it's a property given to the surface itself. Thus, we say that the Euler characteristic is a homeomorphic invariant. Using a formula to relate the Euler characteristic to genus (roughly defined as the number of holes in a surface), we can find out which kind of surface we live in. ■

Important Term

homeomorphic: A notion of equivalence, weaker than isotopic. Two objects are homeomorphic if one object can be cut up into pieces, stretched, and reattached along the cuts to form the other object.

Suggested Reading

Adams, *The Knot Book*.
Richeson, *Euler's Gem*.

Questions to Consider

1. Convince yourself that the Euler characteristic of a genus-3 surface is, in fact, –4. Draw some concrete examples.

2. If two shapes are isotopic, is it easy to show that they are also homeomorphic?

Reaching Boundaries and Losing Orientations

Lecture 12

If you can tell me the surface's orientability—whether it's orientable or not—if you can tell me how many boundary components it has, and if you can tell me what its Euler characteristic is, then you have completely understood everything you need to know about surfaces.

In the last lecture, we saw how the Euler characteristic, which is a local phenomenon, helps us understand a global property—the genus of a surface. We also saw that if 2 surfaces are homeomorphic, they must have the same Euler characteristic. The Euler characteristic of a sphere is 2, of a torus is 0, and of a genus-2 surface, –2. In this lecture, we'll see how to build any surface of any genus by simply gluing together polygons. We'll also learn that there is more to surfaces that just their genus. In particular, we construct surfaces that are non-orientable. These objects have the interesting property that they have only one side. Finally, based on these ideas, we will be able to classify every possible surface imaginable.

In 1884, **Felix Klein**, one of the fathers of topology, developed a method of building surfaces from polygons. We see, for example, how we can construct a torus from a square. In fact, there is a pattern to these constructions, according to which we can build a genus-*g* surface by gluing a 4*g* polygon.

Having used polygons, which have boundaries, to build genus-*g* surfaces, it's natural to consider generalizing the notion of a surface to include things with boundary. Removing interior faces of any surface results in surfaces with boundary components. A classic example is a disc, which we get by removing one face of a sphere. In general, we can give a genus-2 surface 3 boundary components by taking 3 pieces off the surface. Considering this surface, how does the Euler characteristic change with respect to boundary? Every time we remove a face to get a boundary, we lose one value in the Euler characteristic. The Euler characteristic of a genus-2 surface is –2. Thus, the Euler characteristic of a genus-2 surface with 3 boundary components is –2 – 3, which is -5. We can use this information to show that 2 surfaces that are twisted in space differently are actually the same.

Surfaces have another property besides genus and boundary components—orientability. We say a surface is orientable if it has 2 different sides and a surface is non-orientable if it has only one side. A surface is orientable if it has 2 sides that can be painted with 2 different colors, such as a torus or a disc. The Möbius strip is the most famous example in mathematics of a non-orientable surface. Just as we constructed all orientable surfaces by gluing polygons, we can construct non-orientable surfaces by gluing polygons. From the square, we can build the Möbius strip, the Klein bottle (which can only exist in 4 dimensions), and the projective plane.

According to a theorem developed by a number of mathematicians, if we know these 3 properties of a surface—its orientability, its number of boundary components, and its Euler characteristic—we can classify every surface that exists in any dimension possible. ■

Name to Know

Klein, Felix (1849–1925): Klein spearheaded some of the pioneering relationships between algebra and topology. He also showed us how to obtain all surfaces from gluing polygons.

Suggested Reading

Adams, *The Knot Book*.

Richeson, *Euler's Gem*.

Questions to Consider

1. What do you think is the least number of triangles needed to be glued to form a torus?

2. Try to construct a genus-3 surface by gluing the edges of a 12-sided polygon in the right way.

Knots and Surfaces

Lecture 13

What is an algorithm? This is just a machine; you feed this machine your knot projection, and it spits out for you from the machine the surface whose boundary is the knot diagram you gave it. This is Seifert's amazing contribution.

In this lecture, using the classification of surfaces and invariant ideas for knots, we will try to associate surfaces to knots. Given a knot, our goal is to find an orientable surface whose boundary is the knot. For example, we see that if we shade the regions between the strands of a trefoil, we get a 2-dimensional surface with boundary, but orientability seems to depend on the particular projection. Are we relegated to a case-by-case study to answer this question, based on trial and error and considering different projections?

In 1934, mathematician Herbert Seifert came up with an algorithm to create an orientable surface whose boundary is the given knot. Given any knot projection, we begin by orienting the knot. Then, we look at the crossings and, regardless of whether it's an overcrossing or an undercrossing (positive or negative slope), we split it down the middle and pull the crossing apart. Replacing every crossing with a split leaves us with a collection of disjoint circles. We shade each of these and use them as the surfaces to build from. The last step, to go from these 1-dimensional objects to the 2-dimensional object we're looking for, is to attach strips of surfaces to these circles to form the complete surface. If the original crossing was positive, we put a strip that has a positive crossing on it; if the original was negative, we put a strip that has a negative crossing on it. The result is an orientable surface whose boundary is the original knot. We see this algorithm work for projections of both the trefoil and the figure 8 knots.

Based on this procedure, how do surfaces help us better understand knots? Each of our orientable surfaces has one boundary component. If we input a knot projection into the algorithm, the output is an orientable surface with one boundary component, the knot. If we cap off the boundary component, we get an orientable surface without boundary. From the previous lecture, we know that genus or Euler characteristic completely determines this surface. If we can associate to a knot this genus of the surface, it will be a knot invariant. We will have something associated to a knot that's the genus of the surface we get from building the knot. Unfortunately, it turns out that this is not the case. We can, however, say that the *least* genus of any orientable surface bounding the knot is an invariant. Unfortunately again, there are knots for which the minimal genus surface cannot be obtained by the Seifert algorithm for any projection possible.

Here is what the official definition of the genus of a knot is: The genus of a knot is the least genus of any orientable surface bounding the knot.

Earlier, we were unable to prove that the crossing number of knot 1 + the crossing number of knot 2 must equal the crossing number of knot 1 + 2. We can, however, prove the theorem that states that the genus of knot 1 + the genus of knot 2 must equal the genus of knot 1 + knot 2. This theorem answers a question we asked in one of our earliest lectures: Can 2 complicated knots be added together to form the unknot? Can we have subtractions of knots? By this theorem, we are guaranteed that this cannot happen, because the genus of the 2 knots must be added together to get the new genus. If knot 1 and knot 2, somehow, when we added them together, gave us the unknot (genus-0), then the genus of knot 1 and the genus of knot 2 must both be 0 to satisfy the formula. Thus, knots 1 and 2 would have to be unknots themselves. ■

Suggested Reading

Adams, *The Knot Book*.

Richeson, *Euler's Gem*.

Questions to Consider

1. For your favorite knot, use Seifert's algorithm to find an orientable surface whose boundary is the knot.

2. Show that if a knot has genus 1, then it must be a prime knot.

Wind Flows and Currents

Lecture 14

Think of all possible wind flow currents of the Earth right now, flowing and changing and some stopping and moving. … [W]e ask again this question: Does every vector field on the sphere have to have a zero? Does it have to have some place where there's no wind going, or can we have it so wind is flowing everywhere all the time on the sphere?

In this lecture, we'll explore how the currents of wind flow on the Earth's surface, focusing in particular on the following question: At any point in time, is there a place on Earth where there is no wind? We begin with the study of vector fields, fields of movement. Vector fields appear in the world of analysis, which studies change.

A classic illustration of vector fields is provided by the study of a population of owls and mice. A graph shows the mouse population increasing and decreasing in relation to the owl population, with a collection of arrows representing the vector field. In the center of the field is a fixed point that represents stability between the mice and owl populations; this point is the zero of the vector field.

A natural way to get a vector field is the gradient method. The gradient method imagines that something—say, water—is flowing down from the top of the surface. The starting and stopping points of the flow are counted as zeros, places of perfect stability. A number of examples of wind or water flows on a sphere show that the flow is always continuous. Again, we ask the question: Does every vector field on the sphere have to have a zero? Note, too, that there are different kinds of zeros of vector fields, such as the center, gradient, dipole, and so on. How can we measure these zeros quantitatively?

We determine the index of a zero of a vector field as follows: We draw a small disc, a neighborhood, around the zero, then choose the top-most point of this disc and note the vector flow direction of this point. As we move around the circle clockwise in this example, the vector field rotates a full 360 degrees. In general, each time the arrow turns once clockwise, we assign it an index value of 1; each time the arrow turns once counterclockwise, we subtract 1 from the index value. The index measures how wind behaves close to the zero point.

We can also calculate the index by drawing polygons rather than circles around the regions. We place a 1 at the fixed point and a 1 at any vertex of the polygon if the vector fields point inside the polygon at that vertex. We place a –1 along any edge if the vector field points inside the polygon at that edge. We then add the numbers inside the polygon.

For any vector field on a surface *S*, the sum of the indices of all the zeros off the vector field … must be the Euler characteristic. This is a deep relationship on the way flows can move on a surface and is globally governed by the shape of the surface itself.

What happens if we add the indices of all the zeros over an entire surface? Trying several examples, we find that all spheres give an index value of 2, a torus gives 0, and a genus-2 surface gives –2. What we're really finding is the Euler characteristic. This result gives us the Poincaré-Hopf theorem: For any vector field on a surface *S*, the sum of the indices of all the zeros off the vector field must be the Euler characteristic. This shows that the way flows can move on a surface is globally governed by the shape of the surface.

The consequence of this theorem is this: If we're on a sphere with Euler characteristic 2, there must be at least one 0 in any vector field on the sphere since something must be contributing to the index. Alternatively, using our wind flow formulation, there's always a location on Earth with no wind—there has to be—because the sum of the indices must be the Euler characteristic, which is 2. ■

Suggested Reading

Richeson, *Euler's Gem*.

Questions to Consider

1. Can you draw a wind flow on a torus with no fixed points?

2. Can you draw a wind flow on a sphere with exactly seven fixed points?

Curvature and Gauss's Geometric Gem

Lecture 15

Welcome to the greatest theorem, from my perspective, ever discovered: Gauss-Bonnet. It claims that if we know the genus of the surface, we know its total curvature.

In this lecture, we move from the rubber sheet world of topology to geometry, which also studies shapes, but it deals with rigid measurements of distances, angles, and volumes.

Curvature is one of the most visible and defining characteristics of the shape of an object. What we want to do is define curvature in a rigorous way. Consider a circle of radius r. As the radius of the circle increases, the curvature decreases; the 2 values are inversely proportional. Thus, we say that the curvature, K, at a point on a circle is to equal $1/r$. We see that the curvature of a straight line is 0, but what about a general curve in the plane?

Looking at a generic curve, we notice that curvature is defined at every point on the object. To define this, we must associate it to the curvatures of circles. Think of the curvature of a point as the best-fit circle at that point (the osculating circle). Similar to curves, we define curvature of surfaces at a point, but the process is more complicated.

The most famous definition of curvature comes from **Carl Friedrich Gauss**. We can find the Gaussian curvature as follows: We pick a point on a sphere, intersect it with a plane, and we get a circle of radius r with curvature $1/r$; then we cut the sphere 90 degrees to the original cut to get another circle of radius r, which has a curvature of $1/r$. The Gaussian curvature of the point on the sphere is $1/r \times 1/r$, or $1/r^2$.

On a surface, there exists a neighborhood around each point, giving us 360 degrees of freedom to travel from that point. We have infinitely many directions, and each one gives us a curve associated to that surface. Which direction to we pick? Gauss showed that there are 2 special directions to choose at each point on a surface. Looking at some examples, we find that the curvature of a flat piece of paper and that of a cylinder are both 0. This seems counterintuitive, but it's true from an intrinsic perspective. We see that a sphere has positive curvature, a flat sheet of paper has 0 curvature, and a saddle has negative curvature. What does that mean? The curvature shows how things bend. Negative curvature measures pulls in opposite directions and positive measures pulls in the same direction.

Curvature is one of the most visible and defining characteristics of the shape of an object.

What happens if we want to move from curvatures of surfaces at points to the entire surface? We define the total curvature of a surface as the sum of the curvature based on the surface area. For example, the total surface area of the sphere in a global setting is $4\pi r^2$. But the curvature at every point on the sphere is $1/r^2$. To compute the total curvature, we multiply the 2 values. The 2 r^2 values cancel out, and the result is a total curvature of 4π. Notice that this result is not dependent on the radius.

The Gauss-Bonnet theorem tells us that if we know the genus of the surface, we know its total curvature. This is remarkable because genus is a purely topological property, and curvature is a purely geometric property. In particular, the theorem states that the total curvature of an orientable surface, S, is 2π times the Euler characteristic of that surface. ■

Name to Know

Gauss, Carl Friedrich (1777–1855): Known as the Prince of Mathematics, Gauss is considered by many to be the greatest mathematician since antiquity. His foundational work in all areas of mathematics continues to influence our world today. We get the notion of curvature and the powerful Gauss-Bonnet theorem from him.

Suggested Reading

Cromwell, *Polyhedra*.

Devadoss and O'Rourke, *Discrete and Computational Geometry*.

Richeson, *Euler's Gem*.

Questions to Consider

1. Next time you are at a grocery store, consider the curvatures of different types of vegetables, especially kinds of lettuce.

2. Take some Play-Doh and roll it into a ball. As you deform this Play-Doh topologically, notice how the curvature changes. Do you believe the total curvature is fixed no matter how you deform it?

Playing with Scissors and Polygons

Lecture 16

The Bolyai-Gerwien theorem states the following thing: Any 2 polygons of the same area are scissors-congruent. Can you imagine any 2 polygons, no matter how crazy they are, as long as they have the same area, you can take a pair of scissors, cut these into finite pieces, rearrange it, and get the other one? This means that the only quantity to measure scissors-congruence is area.

In this lecture, we move into discrete geometry, which provides us with a world of approximations that we can use to study the natural world. We've already seen that polyhedra approximate spheres. A deep theorem in mathematics says that any surface, no matter how complicated, can always be approximated by a discrete surface.

We begin with a new notion of equivalence of polygons: Two polygons, *P* and *Q*, are said to be **scissors-congruent** if *P* can be cut into smaller polygons such that those pieces can be rearranged to form *Q*. We see a square and a triangle and a square and a cross that are scissors-congruent but are not congruent in the geometric sense. Note that the shapes must have the same area. Scissors-congruence is a weaker form of geometric equivalence, just as homeomorphism was a weaker form of isotopy.

If 2 polygons have the same area, can we always make them scissors-congruent? If not, then what characteristics are we looking for to find scissors-congruency—angle, side lengths? We'll try to build some tools to answer these questions.

We first see that any triangle of some area can be made into a rectangle of the same area using a finite number of cuts. Using a more complicated procedure, we also see that any rectangle of some height and some area can be made into any other rectangle of a different height and the same area. We also need to note here that any polygon can be triangulated. Given any polygon, we can cut it into triangles by drawing a diagonal from one vertex to another.

So far, we've built the following tools: (1) any triangle can be cut up by scissors-congruence to make some rectangle, (2) any rectangle can be cut up by scissors-congruence to make a different rectangle, and (3) any polygon can be triangulated. These tools allow us to prove the Bolyai-Gerwien theorem, which states: Any 2 polygons of the same area are scissors-congruent.

We say 2 polygons, *P* and *Q*, are scissors-congruent—not congruent but scissors-congruent—if *P* can be cut up into smaller polygons where these pieces can be rearranged to form the polygon *Q*.

Given 2 polygons, *P* and *Q*, we cut each polygon into triangles. Next, we convert each of the triangles into some rectangle. Then, we convert each of the rectangles into the rectangle of choice as long as it has the same area and base length 1. We stack up all these rectangles to get a super-rectangle of base length 1, and the height of this super-rectangle is exactly the area of the polygon because area is base times height.

Can we apply scissors-congruence to 3-dimensional polyhedra of the same volume? Unlike polygons, which have one notion of angle, polyhedra have 2 notions of angles. A polygon has only the corner angles, but polyhedra have face angles and dihedral angles (the angle formed between 2 faces). The solution to this question can be roughly stated as follows: If 2 polyhedra, *P* and *Q*, have different kinds of dihedral angles, then they cannot be scissors-congruent. In other words, volume is not enough. We show 2 tetrahedra that have the same base and height but are not the same because they have different kinds of dihedral angles. We could never cut one up and rearrange it to make the other. ■

Important Term

scissors-congruent: A notion of equivalence. Two objects are scissors-congruent if one can be cut up and rearranged into the other.

Suggested Reading

Devadoss and O'Rourke, *Discrete and Computational Geometry*.

Questions to Consider

1. Cut two rectangles of the same area from paper. Try to cut one rectangle and rearrange the pieces to form the other.

2. Can you think of ways to find the least number of cuts needed to make one rectangle scissors-congruent to another?

Bending Chains and Folding Origami

Lecture 17

The world of origami can be broken down into 2 worlds, design and foldability. These are the 2 overarching themes when we think of the word origami and the world of origami.

In the last lecture, we focused on cutting and rearranging polygons; in this lecture, we'll look at folding, which has applications in science, technology, and even business. We begin by looking at folding of 1-dimensional linkages, which are modeled by a rod and joint motion. A powerful use of 1-dimensional linkage by this motion is protein folding, which is the process by which a polypeptide chain folds into a 3-dimensional linkage. For proteins, the correct 3-dimensional folded linkage is essential to their function. In fact, several diseases are believed to result from incorrectly folded proteins.

We focus on a much simpler question based on 1-dimensional linkages: When can linkages lock? A linkage is a collection of rods of fixed length in which the joints are allowed to move. A linkage is unlocked if any configuration of a linkage can be deformed and made into any other configuration. A linkage is locked if there are some positions we can get to in the linkage but we cannot move out of to get to any other position.

A linkage in 3 dimensions is almost like a piece of string that can be moved but from a discrete geometric setting. In 1988, it was proven that linkages can lock in 3 dimensions. We see a proof that a 3-dimensional trefoil knot cannot be unlocked. What about 2-dimensional linkages? Given a 2-dimensional linkage with rods of different lengths, can we always unlock the linkage and move the rods around to get a straight line? This is called the Carpenter's Ruler problem, and it was not solved until 2003. The solution showed that there are no locked chains in the plane.

We're all familiar with origami, the centuries-old art of Japanese paper folding. Let's now define the basics of origami folding in a rigorous way. A fold should be something that's isometric in the sense that it preserves distance. In other words, as we fold, we cannot stretch or tear the paper. Further, a paper cannot self-intersect during the foldings; it cannot pass through itself. A fold on a piece of paper is part of a crease line, and creases may come in 2 forms, mountains or valleys. Visually, valleys are notated by dashes and mountains are notated by dash dots.

Origami can be broken down into 2 worlds, design and foldability. Origami design concerns itself with folding a given piece of paper into something having a particular shape. Foldability concerns itself with asking which crease patterns can be folded into an origami pattern that uses exactly the creases on the pattern. And how do we know which patterns will succeed and which ones will not be able to be folded flat without new creases? According to the Maekawa theorem, a necessary (but not sufficient) condition is that the number of mountain and valley folds around each vertex must differ by 2.

… how do we know which ones we can succeed at and which ones will not be able to be folded flat without new creases? There's a classic result called the Maekawa theorem of 1989. It says that the number of mountain and valley folds around each vertex must differ by 2.

A powerful use of origami folding comes in the form of the one-cut conjecture. Think of drawing a polygonal shape on a piece of paper. Is it possible to fold the piece of paper, make one straight cut, and cut out the polygonal shape? The solution depends on lining up all the edges on one line, which of course, is based on folding. ■

Suggested Reading

Devadoss and O'Rourke, *Discrete and Computational Geometry*.

Lang, *Origami Design Secrets*.

Questions to Consider

1. Using a rectangular sheet of paper, make a grid design with random mountain and valley folds. Can you flat-fold this paper? Do this for other random designs.

2. Draw your favorite polygon on a piece of paper and try to cut it out using foldings and one straight cut.

Cauchy's Rigidity and Connelly's Flexibility
Lecture 18

This results in Arthur Cauchy's rigidity theorem from 1813, one of the most beautiful theorems of polyhedra. It says the following thing: If 2 closed convex polyhedron are combinatorially equivalent—in other words, if they have the same gluing information with congruent faces, the same pieces of the puzzle—then the 2 polyhedra must be identical.

In this lecture, we move from origami and folding of objects to the opposite spectrum, the rigidity of objects. We are motivated by a question of rigidity related to stereoisomers. Recall that stereoisomers are molecules that have the same basic arrangement of atoms and bonds but differ in the way the atoms are arranged in space. There are several different stereoisomers that come from the same set of pieces, the atoms, and their gluing information, the bonds. An example is the dichloroethene molecules, C2H2Cl2, made of carbon, hydrogen, and chlorine. Although they're made up of the same pieces of the puzzle and we're asked to glue the puzzle the same way, the resulting objects are different. And because they are not identical, they have different properties. Form and function are once again related. We'll prove that if stereoisomers form **convex** polyhedra, they must be congruent.

We begin with an understanding of **Cauchy's** rigidity theorem, which states: If 2 closed, convex polyhedra are combinatorially equivalent—in other words, if they have the same gluing information with congruent faces—then the 2 polyhedra must be identical. This theorem has stunning implications for stereoisomers. For a collection of stereoisomers that close up to a convex polyhedral structure, there is only one kind of object that can be made from a chemical perspective. Further, if there is only one way to make this convex object based on the gluing information and the congruent faces, all the dihedral angles of the 2 polyhedra must be the same, which means that they are rigid.

The proof of Cauchy's rigidity theorem is quite ambitious. It is an amazingly complicated result using 2 lemmas and a beautiful method of going back and forth between topology and geometry and converting dihedral angles into angles of polygons. We go through the proof, starting from the assumption that we can find a contradiction to Cauchy's rigidity theorem, but we learn we cannot.

We know that convex polyhedra are rigid, but what about non-convex polyhedra? Can they flex? This remained an open question from the early 1800s until 1976, when Herman Gluck proved that rigid polyhedra are everywhere, although he didn't prove that all polyhedra are rigid. In 1978, Robert Connelly proved that flexible polyhedra exist; he constructed one having 30 triangular faces. Currently, Klaus Steffen has reduced this polyhedron to 14 triangles. This work prompted a new question: As we flex the polyhedron, does its volume change? Does it form a bellows? In the 2-dimensional case, the area changes as the polyhedron flexes, but it has been proven that the volume remains fixed in the 3-dimensional case.

It is … good sometimes to actually look under the hood of some of these theorems, as we did today, to see how mathematicians think and what makes things work.

In the next lecture, we push further forward into geometry by polygons and their uses in terrain reconstruction data. ■

Name to Know

Cauchy, Augustin-Louis (1789–1857): A powerhouse in analysis and a prolific writer, who gave us the arm lemma and the rigidity theorem for polyhedra.

Important Term

convex: An object is convex if the line segment containing any 2 points in the object is contained within the object.

Suggested Reading

Cromwell, *Polyhedra*.

Devadoss and O'Rourke, *Discrete and Computational Geometry*.

Questions to Consider

1. In what other places in nature and in architecture do you see flexible edges but rigid plates?

2. Try to construct a flexible polyhedron yourself. Why is it difficult to do?

Mountain Terrains and Surface Reconstruction

Lecture 19

Can you start at any vertex, a triangulation, and go to any other vertex, another triangulation, by just flipping, going through these edges? A beautiful theorem, by Lawson in 1971, says that this is indeed true. Again, this is a powerful theorem since we can make global changes to my triangulation by these small local moves.

This lecture focuses on using polygons to approximate large-scale terrains on the Earth. Satellites and airplanes are used to gather sample clouds of height data for given regions. Our job, using these data, is to try to reconstruct the terrain based on the observed values. These reconstructed terrains are called meshes and can be applied to surfaces that are important in other applications, such as medical imaging and facial recognition software.

Assume we're given a collection of points in space. First, we project the 3-dimensional point data into the 2-dimensional plane. Next, we triangulate the plane, forming a mesh. Finally, we lift each of the resulting triangles up to its appropriate height. Note that the terrain we get in 3 dimensions is heavily influenced by, and linked with, the triangulation we chose in 2 dimensions.

We see some examples that show we can always construct a triangulation given any point cloud on the plane. A triangulation is defined as a subdivision of the plane by a maximal set of non-crossing edges. Triangulations can be constructed by the incremental method and numerous other methods. The number of triangles we get in any given triangulation depends on the number of points in the cloud and the number of points that are on the boundary. This theorem tells us that any triangulation of a given point cloud will have the same number of triangles, but this number might be quite large.

We need to study a superstructure, a structure larger than what we're talking about, that keeps track of the world of all triangulations of a point cloud. We find this in looking at the flip graph of all the triangulations of a point cloud. A flip is defined as removing a diagonal from a convex quadrilateral and replacing it with the opposite diagonal. On a flip graph, each vertex corresponds to a possible triangulation, and we show that we can move from one triangulation of a point cloud to any other triangulation just by flipping diagonals one at a time. This is a powerful result because it illustrates global changes made to a triangulation by local moves. The flip graph is an example of a discrete **configuration space**. It is an object that keeps track of all possibilities, yet it's discrete because it has vertices and edges.

… every triangulation of a point cloud *S*—if it has *i* points in the interior and *b* points on the boundary—has exactly 2 *i* + *b* – 2 triangles.

Returning to terrain reconstruction, we consider an example of a point cloud showing data gathered from a mountain range. We see 2 possible triangulations of these point cloud data, one of which intuitively seems more accurate than the other. In fact, the triangulation we chose has "fat" angles, which are the ones we want. We want to avoid "skinny" angles. To measure how good one triangulation is compared to another, we compare their angle sequences. Our dream triangulation is called the Delaunay triangulation, which is the one with the largest angle sequence. We can find the Delaunay triangulation by using the flip graph.

We've been looking at point clouds in 2 dimensions, but in computational geometry, the focus is on point clouds in 3-dimensional space. Here, we connect points, not to get triangles on the plane, but tetrahedra in 3 dimensions, called tetrahedralizations of point clouds. ■

Important Term

configuration space: The space that keeps track of all possible ways an object (or a collection of objects) can be arranged.

Suggested Reading

Devadoss and O'Rourke, *Discrete and Computational Geometry*.

Questions to Consider

1. Given any point set with n points where k points are on the hull, show that any triangulation of the point set has exactly $3n - 3 - k$ edges

2. Find another procedure or algorithm that finds a triangulation of a point cloud.

Voronoi's Regions of Influence

Lecture 20

What we are seeing are Voronoi diagrams, partitions of space based on regions of influence. Each region kind of emanates its influence around it and is governed by a source.

In the last lecture, we considered point clouds as coming from terrain data. In this lecture, we'll explore a different view of these points. If the points represented hospitals in a city, for example, how could we find the closest hospital? The key is to look at this problem from the perspective of the hospital and discover which region of the city it serves. In mathematics, these regions of influence emanating from 2-dimensional points on the plane are called Voronoi diagrams, and they have a powerful influence in the real world.

Once again, we start with a point cloud in the plane, but now these points represent sources of influence that govern the growth of our Voronoi diagrams. An example of such a diagram shows 5 points, each of which controls a region. The Voronoi edge is where 2 points meet, and the Voronoi vertex is where 3 edges meet. Note that we are dealing with 2 kinds of points. We have the original points from the cloud, and once we have these points, we have an underlying, invisible Voronoi structure.

A classical result from geometry implies that each Voronoi region, each region of influence, must be convex. Knowing that and given a point cloud, how can we construct the Voronoi diagram? We use an incremental approach, starting with the idea that we have an existing Voronoi diagram to which we need to add one point, *p*. Our method leaves everything in the plane alone except for the region right around *p*. This method also allows us to build a new Voronoi diagram with as many points as we're given.

What do we need to do to count the number of Voronoi vertices, these invisible vertices that tell us where 3 regions meet? What about the number of Voronoi edges? Are they based on position; are they based on the angles they form with one another? Will the fact that these regions are convex help us somehow? We show that for n points in the cloud, the number of Voronoi vertices is $2n - 5$ and the number of Voronoi edges is $3n - 6$.

The Voronoi diagram gives us information about which points in the cloud are closer to one another. If we just start drawing lines between points with adjacent Voronoi regions, then straighten out the lines, we end up with triangles. Thus, the point cloud, using the Voronoi diagram, gives us a new triangulation, which in turn, gives us information about closeness. Interestingly, the Delaunay triangulation, used for terrain reconstruction, as we saw in the last lecture, is the same triangulation we get that encapsulates data about proximity and closeness of points.

We close the lecture by looking at a problem that has remained unsolved after 500 years. Is it possible to cut a convex polyhedron along its edges and unfold it flat, with all the pieces connected? Every example that mathematicians have ever tried has worked, but we still don't know why. We can get a partial solution by changing the problem to allow ourselves to cut not just along the edges but also along the faces of the polyhedron and by using a Voronoi diagram to give us information about equidistant points. ■

Suggested Reading

Devadoss and O'Rourke, *Discrete and Computational Geometry*.

Questions to Consider

1. Place 10 random points on a piece of paper and draw its corresponding Voronoi diagram.

2. Now place these 10 points in special positions, where they all lie on a line, or on a circle, or on another geometric design. How do the corresponding Voronoi diagrams change?

Convex Hulls and Computational Complexity

Lecture 21

This Big O measures how fast an algorithm works. But the speed is not based on actual time … but rather the number of steps needed to finish your algorithm.

In this lecture, we look at a more basic feature of a point cloud. We want to find the smallest convex region that contains the points. While thinking about this problem, we venture a bit deeper into the mind of a computer scientist.

Given a point set, its **convex hull** is the smallest convex set that contains the point set. Recall that a set is convex if the line segment between any 2 points in the set is contained in the set. The best way to visualize a convex hull is to imagine hammering pegs (points) into the plane. Then, take a giant rubber band containing all the points and let it go. It snaps onto the convex hull. In other words, given this point set, the convex hull is the rubber band that snaps around these points. The convex hull of a point set must be a convex polygon. Convex hulls are extremely useful for narrowing down a large pool of data to get a rough boundary structure.

To find the convex hull, we have to define "smallest." A smallest convex set is contained in every convex set that contains the cloud. Thus, the convex hull is the intersection of all convex sets containing the given point cloud. The following are equivalent to the convex hull of a point set: The convex hull is a convex polygon with the least area that contains the given point cloud. The convex hull is also a convex polygon with the least perimeter that contains the given point cloud. Mathematicians can come at the question from a theoretical perspective in terms of intersections of sets. For computer scientists, the key is to find the polygon defined by the rubber band. It's only the key vertices, called the hull points or hull vertices, from the point cloud that a computer scientist is interested in.

We can use an incremental algorithm to find the convex hull, but that method isn't very efficient. In computational convexity theory, the Big O measures how fast an algorithm works, but the speed here is not based on time; it's based on the number of steps needed to finish the algorithm. Adding 2 numbers, for example, requires only one step; thus, the Big O is just 1. The Big O for finding a number in a list of *n* numbers is *n*. For sorting numbers in a list, the Big O is *n*log*n*. Note that the log of *n* is between 1 and *n*. For the incremental method of finding the convex hull, the Big O is *n*2.

Triangulations are useful; we talked about that. Voronoi diagrams are useful; indeed, we talked about that. But why would we care about convex hulls? They are extremely useful if you're interested in fast data processing.

Some convex hull algorithms use geometric insight. The gift-wrapping algorithm, for example, is based on the geometry of angles, and its speed is $n \times h$, where *h* is the number of points in the hull. However, if we don't know the configuration of points in the cloud, then the gift-wrapping algorithm would be just as inefficient as the incremental method. The Graham scan algorithm is similar to the gift-wrapping algorithm in that in uses the bottom rightmost point of the cloud as an anchor and scans the points, looking at the angles they form. In this algorithm, the points are sorted based on the angles they form, and the hull is constructed based on this sorted list. The speed of this algorithm is *n*log*n*. As it turns out, this is the fastest algorithm for finding the convex hull. ■

Important Term

convex hull: Given a point cloud, its convex hull is the smallest convex set containing the point cloud.

Suggested Reading

Devadoss and O'Rourke, *Discrete and Computational Geometry*.

Questions to Consider

1. Place 10 points in special positions, where they all lie on a circle or another geometric design. Which convex hull algorithms are the fastest for each of these special positions of points?

2. Think of how to extend the incremental algorithm or the gift-wrapping algorithm to point clouds in three dimensions.

Patterns and Colors

Lecture 22

Can we get a 4-color theorem? That is a dream of dreams. That means the local condition and the global condition are actually the same. Amazingly, after 150 years, a revolution in mathematics occurred—this is, indeed, true. A 4-color theorem exists.

In this lecture, we move from geometric tilings to topological tilings. As we know, the tilings and patterns of animals and plants are filled with color and design of almost infinite possibilities. In this lecture, however, we learn that although complexity knows no bounds, simplicity has natural mathematical limitations.

We have talked about the geometry of partitions of the plane, but what can we say about the topology? We have Euler's formula, which gives the relationship among edges, vertices, and regions in the plane. Can we say more, especially when dealing with colors and patterns of regions? In this lecture, we ask: Given a collection of patches on the plane, what is the least number of colors needed so that 2 patches that are next to each other must have different colors?

We consider particular examples of this problem, focusing on one region and looking at its surrounding patches. Remarkably, only 4 colors are needed to satisfy the conditions of the problem. Note that this problem is a topological issue, not a geometrical one. It involves relative position rather than area, length, or size. Historically, this problem dealt with coloring countries on a map, and it gave rise to the 4-color conjecture, which states that any map can be colored using just 4 colors with no 2 adjacent countries sharing the same color. Underlying this issue is the shape—the topology—of adjacency.

We prove the following lemma by contradiction: Every map we can imagine must have at least one country with 5 or fewer neighbors around it. Using this lemma, we get the 6-color theorem, according to which every map can be colored with 6 colors. Again, we prove this theorem by contradiction, but can we do better? We used Euler's formula to prove the 6-color theorem, but we need to look at chains of colors in the plane to prove the 5-color theorem. Once again, we see this proof by contradiction.

In proving the 5-color theorem, we notice that only 4 colors were needed in the local condition; is that true for the global condition? Are the local and global conditions actually the same? This question was solved by Kenneth Appel and Wolfgang Haken using a supercomputer, but their result did not provide a deep understanding of the mathematical ideas behind it. More than 20 years later, a group of 4 mathematicians provided a much simpler proof that satisfies almost all of their colleagues.

Notice that this problem about coloring that we saw from this local perspective is an issue with topology, not geometry. It does not have to do with area, length, or size; it has to do with relative position.

All the issues we've been thinking about related to coloring patches on the plane are identical to issues on the sphere. What about a torus? Recall that the Euler characteristic for the torus is $v - e + f = 0$. Using this, we can show that every map on a torus has at least one country with 6 or fewer neighbors, and we can use the same ideas that we used earlier to show that, at most, 7 colors are needed to color the map on a torus. Unlike our work with the plane, however, we cannot reduce the number of colors for a torus. In general, we have a formula for determining the number of colors needed for surfaces of genus g. For example, the formula tells us that 8 colors are needed for a genus-2 surface, 9 for a genus-3, and so on. Further, we know that we cannot reduce the number of colors needed for these surfaces. ■

Suggested Reading

Weeks, Jeffrey. *The Shape of Space*.

Questions to Consider

1. Draw a random map with numerous countries. Try to color this map using only four colors.

2. Draw a random map on a torus. Try to color this map using only eight colors. Can you do it with fewer colors?

Orange Stackings and Bubble Partitions

Lecture 23

Is the grocer packing the most efficient way of packing spheres in space—as long as all the spheres are identical? It's a simple question. It requires some of the most sophisticated mathematics ever created.

We've touched on packaging a number of times in these lectures; in this one, we focus not on the efficiency of the package itself but on the packing of goods inside it. This topic relates to optimization, that is, getting the most out of a given situation.

In 1611, **Johannes Kepler** asked: What is the most efficient way to pack identical spheres as tightly as possible in space? Kepler believed that the best arrangement is the one used to stack pyramids of oranges in the grocery story, called the grocer's packing.

In looking at a 2-dimensional version of Kepler's conjecture, we see a couple of possible arrangements of discs on the plane, including the square lattice packing and the hexagonal lattice packing. Efficiency of packing compares how much area the discs cover compared to the total area they could possibly cover. This mathematical measurement is called density and is given as a percentage. For the greatest efficiency, we need the greatest possible density. The density of the square lattice packing is about 78.54% and that of the hexagonal packing is 90.69%. It has been proved that the hexagonal packing in general—compared to any random tiling—is best.

Moving into 3 dimensions, we again consider some density calculations. With balls arranged in a cubicle lattice structure, we get a density of 52.7%, while the grocer's packing has a density of about 74.05%. Kepler showed that the density and the structure of the square lattice packing and the hexagonal packing are the same in 3 dimensions. In 1830, Gauss showed that the grocer's packing was the best lattice packing.

What if we had just a random collection of oranges rather than a perfect lattice structure? **László Fejes-Tóth** paved the way for thinking about this problem by focusing on Voronoi regions rather than the spheres themselves. With his approach, the problem became one of finding maximal density in a world with 150 dimensions. Around 1995, **Thomas Hales** combined Delaunay tetrahedralizations and Voronoi regions to attack the 150-dimensional problem and finally solved the Kepler conjecture.

Finally, we turn to another question posed by Kepler: Why are honeycombs hexagons? This requires converting the problem of packing into a problem about partitions: What is the most efficient way to partition space into equal volumes? Hales, again, proved that the hexagonal structure of a honeycomb is the most efficient in 2 dimensions, but earlier, Fejes-Tóth had shown a slightly more efficient structure in 3 dimensions. Whether or not this is the most efficient structure is still an open question.

But there is something in the Weaire-Phelan structure's favor to make their structure the candidate for optimal partitioning of the space: This structure appears in nature.

What about a 3-dimensional partition of space itself? Can all of space be broken up into 3-dimensional pieces with equal volume while minimizing surface area? This is called the **Kelvin** problem. In 1993, 2 mathematicians had the breakthrough idea of using 2 shapes of equal volume, a dodecahedron and a new polyhedron, to tile space. Two of the dodecahedra and 6 of the new polyhedra form a superstructure called a cluster, which can be used to partition space beautifully, but it's not known whether this structure is the most efficient. ■

Names to Know

Fejes-Tóth, László (1915–2005): One of the fathers of modern discrete geometry, his works influence practically all areas of this field today. In particular, he investigated packings and partitions and laid the framework for understanding the Kepler conjecture, later solved by Hales.

Hales, Thomas (1958–): Hale solved two powerful open problems (some of the oldest ones in discrete geometry): the Kepler conjecture and the honeycomb conjecture. He believes that partnership with computers will be fundamental in solving future mathematical problems.

Kelvin, William Thomson, Lord (1824–1927): A powerful scientist, Kelvin had wonderful notions of shape and nature. He believed that knots embodied properties of atoms and worked with soap bubbles to posit an efficient tiling of space. He is best known for his Kelvin temperature scale of absolute zero.

Kepler, Johannes (1571–1630): A mathematician and astronomer, Kepler tried to relate platonic solids to the solar system. He also made a conjecture about the best way to stack spheres in space. He is most well known for discovering the elliptical motions of planets around the Sun.

Suggested Reading

Aste and Weaire, *The Pursuit of Perfect Packing*.

Questions to Consider

1. Can you come up with other shapes that could be potential chambers where bees can live? Remember, these need to stack next to each other and back to back.

2. Try to measure the density of the hexagonal Kepler orange stacking.

The Topology of the Universe

Lecture 24

For us to understand the Earth's surface, we can have an extrinsic perspective by leaving the surface. But to understand the universe, we have to have an intrinsic perspective because we are in it. We don't have the power to leave it.

In this lecture, we launch ourselves into the 3-dimensional realm and try to understand the shape of the universe itself. We know, of course, that the Earth is a sphere; it is finite, but it has no boundary. We usually think of the universe as infinite, but why must it be any different from Earth? What are the possible shapes of the universe, and how could we discover them? In looking at the Earth, we are in an extrinsic setting in 2 dimensions (the surface of the Earth), but in looking at the universe, we are in an intrinsic setting in 3 dimensions.

Let's start with a mathematical perspective. A 3-dimensional object is called a 3-manifold. Every point in the manifold is surrounded by a 3-dimensional region of freedom. Manifolds can be finite or infinite in volume; they may or may not have boundary; and they may or may not be orientable. For our purposes, we will consider manifolds that are finite in volume, that are without boundary, and are orientable.

We can construct a 3-manifold with multiplication that uses shapes rather than numbers. A line segment times a line segment gives us a square. A circle times a circle gives us a torus. When we multiply, the dimension of the new object becomes the sum of the dimensions of the multiplied objects. Because each circle is 1-dimensional, 2 circles multiplied becomes 2-dimensional, and 3 circles multiplied must be 3-dimensional. The result is called a 3-torus, which is our first 3-manifold that satisfies the conditions for a possible shape of the universe. If we multiply a circle times a sphere, we get a thickened sphere, another 3-manifold that satisfies our conditions.

We cannot build all surfaces or all 3-manifolds from multiplication. Recall, however, that we could glue polygons together to build all surfaces that have a finite area, are without boundary, and are orientable. Can we glue polyhedra together to get 3-manifolds? We have already seen the 3-torus, built from gluing sides of the cube. We can also construct the one-half twist cube space, the one-third twist hexagonal space, and the Seifert-Weber space. Indeed, every 3-manifold comes from gluing sides of a polyhedron. Mathematician **Jeffrey Weeks** created software called *Curved Spaces* that allows us to see what such worlds look like.

What would it feel like to be inside this 3-dimensional torus? Notice that the copies of this galaxy that you see are actually the same galaxy. As we walk around and fly inside this 3-dimensional torus, we get a sense of vastness. [Said in relation to the *Curved Spaces* software.]

Given that we cannot leave the universe, how can we ever know its shape? The answer lies in cosmology with cosmic crystallography. If we lived in a manifold built from gluing faces of a polyhedron, we would see copies of ourselves in different parts of space. If we understand the gluing based on the copies we see, then we can understand the manifold we live in. There are 2 problems with this answer, however. First, it's hard to see such patterns in our Milky Way Galaxy because we are not looking at our own galaxy head on but from different angles. Second, the light that reaches us is billions of years old; thus, the copies we see are from billions of years ago. The study of cosmic microwave background radiation, however, can bypass these problems and give us a beautiful visual map of the universe. ■

Name to Know

Weeks, Jeffrey (1956–): A student of Bill Thurston's, Weeks uses his understanding of 3-manifolds to study the shape of the universe. He became a MacArthur Fellow in 1999 and wrote the program *Curved Spaces*.

Questions to Consider

1. Download Jeffrey Weeks's *Curved Spaces* program. As you fly through different worlds given by gluing polyhedra, note your experiences.

2. Can you think of a creative way to describe the world we get when we multiply the shape "Y" by itself?

Tetrahedra and Mathematical Surgery

Lecture 25

> **… we could not get all 3-manifolds from multiplication, but we could get them from gluing polyhedral. … This gluing, along with understanding cosmic microwave background radiation, could be the key to find the shape of the universe from our intrinsic perspective inside this 3-manifold called the universe that we live in.**

We classified surfaces based on their genus, their boundary, and their orientability. As we will see in this lecture, we have 3 ways of building manifolds, but the question of whether a classification theorem exists for manifolds remains open.

We begin with one of the most important mathematical properties of 3-manifolds, tetrahedralization. In the same way that every surface can be triangulated, every finite-volume 3-manifold can be tetrahedralized. This result has powerful consequences; it enables us to prove, for example, that every finite 3-manifold comes from gluing sides of some polyhedron. If we start gluing tetrahedra, we get a ball that has triangular faces as its boundary. Gluing these faces to another structure based on the original tetrahedralization structure gives us a 3-manifold.

For 2 dimensions, triangulations led to the Euler characteristic. For 3 dimensions, we have the Poincaré-Euler characteristic, a generalization of Euler's idea for surfaces to arbitrarily high dimensions. The generalized Euler characteristic is $v_0 - v_1 + v_2 - v_3 + v_4$ all the way up to v_n, with alternating signs. This is a topological invariant. The Euler characteristic we've been studying for surfaces extends to any dimension. The Euler characteristic is powerful for surfaces, but it does not appear to be strong for 3-manifolds. For both the 3-dimensional sphere and 3-dimensional torus, the Euler characteristic is 0. To understand this result, we need to build manifolds from a fresh perspective.

Recall that we glued 2 balls together to get a 3-dimensional sphere. Each solid ball was a 3-manifold with boundary, but we removed the boundary by gluing them together. What happens if we glue 2 solid tori together instead of 2 solid spheres? Note that a torus has a meridian curve and a longitudinal curve. We can also draw another curve in the torus called the *PQ* curve. A *PQ* curve is one that wraps in the meridian direction *P* times and the longitudinal direction *Q* times. We can use these curves to perform our gluing, and as a result, we get different 3-manifolds.

It turns out that the Euler characteristic we have been looking at is but a shadow of a deeper reality coming from the world of algebraic topology—algebra and topology meeting.

The idea of gluing 2 solid tori can be generalized further. In fact, every orientable 3-manifold with finite volume can be constructed by gluing 2 solid surfaces called handlebodies. Alternatively, we can say that every 3-manifold contains within it a surface of genus *g* such that cutting out along this surface results in 2 handlebodies of the same genus. This structure is called a Heegaard splitting. We prove this theorem using our knowledge of the tetrahedralization of 3-manifolds.

The Heegaard splitting is exactly what we need to prove a surprising result about the 3-manifolds that meet our conditions: All 3-manifolds that are finite and orientable and have no boundary have Euler characteristic 0. In other words, although the Euler characteristic is an invariant, it is a useless invariant in 3 dimensions.

To close this lecture, we introduce one final way of constructing all 3-manifolds, called **Dehn surgery**. The beauty of Dehn surgery is that it relates 3-manifolds to 1-dimensional knots and links. In general, it can be shown that every 3-manifold comes from Dehn surgery on links in the 3-sphere. ■

Name to Know

Dehn, Max (1878–1952): A student of David Hilbert, Dehn is known for his work in geometry and topology, particularly Dehn invariants for scissors-congruence of polyhedra and Dehn surgery for manipulating 3-manifolds.

Important Term

surgery: The process of cutting and regluing 3-manifolds.

Suggested Reading

Adams, *The Knot Book*.

Questions to Consider

1. Find a tetrahedralization for the 3-torus or a 3-manifold of your choice.

2. Find a Heegaard splitting for the 3-torus or a 3-manifold of your choice.

The Fundamental Group

Lecture 26

[In] one dimension, we thought about isotopy; 2 dimensions, we thought about homeomorphism; and in 3 dimensions, we're going to think about homotopy.

This lecture looks at a useful invariant of manifolds, the fundamental group. Given a manifold, we associate to it an algebraic group in such a way that this group is an invariant of the manifold. Recall that a group is a set of elements with a composition map that has 3 properties: associativity, an identity element, and an inverse element. Our first goal is to associate elements to a surface that we can multiply together to get a group structure. This new group structure is one of the most important ideas in topology.

The Euler characteristic didn't help us in differentiating 3-manifolds. For this task, we need a new kind of equivalence, **homotopy**, which can best be defined as continuous deformation. Here, in addition to pulling and stretching, we're allowed to compress, which changes dimension itself. Under homotopy, a 2-dimensional disc and a 3-dimensional ball are equivalent to a zero-dimensional point.

The elements of the group we associate to a surface, called the fundamental group, are loops drawn on the surface. We consider a disc and some loops to see that they are homotopically equivalent. The loops can be pulled and stretched, as in rubber sheet geometry; they can also be pulled into a point and are allowed to have self-intersections.

To attack our problems of groups, we need this idea of homotopy, along with the elements of the group and the composition map. First, we choose any point on our surface as a fixed base point. The elements of the group are loops that start and end at the base point. These loops have an orientation. Given that they start and end in the same place, it makes sense to add one loop to another because they all have the fixed base point in common. If we can continuously deform one loop into another, these homotopically equivalent loops constitute one element in our group. In other words, when we talk about elements, we are talking about a collection of loops that are all homotopically equivalent. Another loop that cannot be continuously deformed into the first is a different element.

Given our elements, we need to check identity, inverses, and associativity to ensure that they actually form a group. The identity loop is essentially just a point; it remains at the base point. The inverse of a loop is simply the same loop except in opposite orientation. Finally, it can be shown that loops $(ab)(c) = (a)(bc)$. We now have a group structure called the fundamental group. The elements of the group are loops on surfaces up to homotopy equivalence, and the composition map is just following one loop, then following the another one all the way to the base point. Note that the fundamental group does not have the property of commutativity.

What we are about to construct, this new group structure, is one of the most important ideas in topology and one of the most intricate and complicated we have encountered in these lectures.

We can now calculate the fundamental groups of some surfaces. The fundamental group of the sphere turns out to be trivial. For integers, the number 1 generates the whole group; in a similar way, the simple loop around the annulus generates every element of the fundamental group. The fundamental group of the torus is generated by 2 letters, α and β, where α is the loop around the meridian and β is the loop around the longitude. For the torus, these 2 letters commute; that is, α and β can be interchanged freely. ■

Important Term

homotopic: A notion of equivalence, weaker than homeomorphic. This notion deals only with continuous deformations where self-intersections are allowed.

Suggested Reading

Hatcher, *Algebraic Topology*.

Questions to Consider

1. Group the letters of the English alphabet into homotopically equivalent classes.

2. Convince yourself that the elements of the fundamental group of the torus commute.

Poincaré's Question and Perelman's Answer

Lecture 27

We finally are able to solve a problem that has plagued us since our earliest lectures—to get a complete knot invariant—all for the sake of converting it to another problem—comparing knot presentations … which happens to be unsolvable.

To distinguish between 3-manifolds, we created the fundamental group, an algebraic structure used to measure shapes. The elements of the fundamental group are loops under a concept of equivalence called homotopy. This gave us a group structure as an invariant—the shape itself—but the fundamental group is not a complete invariant for 3-manifolds. Nonetheless, it is very useful in several situations.

Let's begin by looking at knot complements; again, think of these as knots bored out of the 3-dimensional sphere. Earlier, we saw that a knot is completely determined by its complement. Indeed, this 3-manifold, the knot complement, is a complete invariant of knots. And we can use the fundamental group to study knot complements in order to study the knots themselves. Given a knot, the fundamental group of its knot complement is the complement of the knot, called the knot group.

We can compute the knot group using a method developed by Wilhelm Wirtinger. This method is based on the knot projection alone. In looking at this method of constructing the knot group from the projection, remember that a group is a collection of letters, called generators, from which we can create the elements of the group, which are the words. We can create all possible words from these generators. Further, we have a collection of rules, called relations, that explain which words are equivalent to which other words. A set of generators and relations for a group is called a presentation of a group.

With Wirtinger's approach, we first orient the projection of the knot. Second, each strand of the knot projection gets a generator. Third, each crossing gives us a relation. Given each strand as a generator and with each crossing giving us a relation, we get the presentation. As an example, we calculate the Wirtinger presentation of the figure 8 knot group. The procedure we use works for any knot. Further, Wirtinger proved that different projections of the same knot will give us the same group; thus, this is a knot invariant. Note, however, that while the groups are the same, their presentations might look different. And given 2 presentations of 2 groups, there is no method to determine whether both of them are describing the same group. Thus, although the knot group is a complete invariant for prime knots, it becomes practically useless.

Our dreams have come true. … Our desire [has been] to find a complete invariant of knots, and now we have one. The complement of the knot, this 3-manifold, is a complete knot invariant.

In the last lecture, we saw that the fundamental group of the 3-dimensional sphere is the trivial group; it has the identity element in it and nothing else. Understanding the 3-sphere might seem easy, but it has launched the greatest problem in topology in more than 100 years, proving the **Poincaré** conjecture, which is: The only 3-manifold with trivial fundamental group is the 3-dimensional sphere. In the 21st century, mathematicians began to look at this conjecture from different dimensions. The conjecture was proved for the 2-dimensional case, for all dimensions higher than 5, for the 4-dimensional case, and finally, in 2006, for the 3-dimensional case. Mathematicians are now working on a simpler answer to the Poincaré conjecture. ■

Names to Know

Perelman, Grigori (1966–): Perelman completed the work of Richard Hamilton, using Ricci curvature flows to prove the Poincaré conjecture and, most likely, Thurston's geometrization conjecture itself. Although he won the Fields Medal in 2006, he did not accept it.

Poincaré, Henri (1854–1912): One of the greatest and most prolific mathematicians in history, Poincaré worked in geometry, algebra, number theory, physics, and the philosophy of science. He is credited with being the father of modern topology.

Suggested Reading

Hatcher, *Algebraic Topology*.
O'Shea, *The Poincaré Conjecture*.

Questions to Consider

1. Choose a projection of the trefoil and calculate the fundamental group for it using the Wirtinger presentation.

2. Choose another projection of the trefoil and find its Wirtinger presentation. Can you show that both of these presentations are indeed of the same group?

The Geometry of the Universe

Lecture 28

As we move closer to the speed of light, time itself slows down from our perspective. With this one beautiful idea, Einstein forever destroyed the notion of absolute rest and absolute time.

We now enter the world of 3-dimensional geometry, moving from Isaac Newton's ideas of geometry to those of Einstein and his theories of relativity and cosmology. Around the mid-1800s, **Bernhard Riemann** gave a mathematical lecture that introduced the idea of separating the topology of a space from the geometry on that space. Riemann argued that distance must be defined independently of the space in which you live. In other words, we can live in identical topological spaces but have completely different geometric experiences based on how distance is defined in each of our spaces. This notion of distance is called a metric.

With the classic Euclidean metric, the distance between points *a* and *b* on a plane is given by the distance formula. With the Manhattan metric, the distance between the same 2 points is given by a different formula. We have kept the underlying topology but changed the geometry; as a result, the concept of the shortest path between 2 points (called a geodesic) has changed. Because light travels along the shortest path between 2 points, if we change the metric, we change how light travels.

The geometry of 2 dimensions is determined by the sum of the angles of a triangle. No matter how the metric is defined for surfaces, we get 3 possible geometries—Euclidean, spherical, and hyperbolic. In the 1970s, **William Thurston** showed that there are only 8 geometries in 3 dimensions.

How do metrics and geometries in the world of 3-manifolds appear in our universe? The first intersection of geometry with our universe comes in the form of Einstein's special relativity. In rethinking the meaning of time, Einstein solved the paradox between Maxwell's idea that light is never at rest and Newton's theory of motion. Einstein claimed that everything moves in relation to other things. In particular, light speed is fixed and all other speeds are relative to light. Indeed, absolute time does not exist. As we move closer to the speed of light, time itself slows down from our perspective. With this one idea, Einstein forever destroyed the notion of absolute rest and absolute time. We can no longer separate space and time as 2 distinct spaces. There's nothing called space and time separately but one unified shape called spacetime. The topology of spacetime is something like a genus-3 surface, where space and time cannot be separated.

Einstein introduced the theory of general relativity to address the disconnect between special relativity and gravity. He realized that the geometry of spacetime is curved. He also showed that the mass of an object causes a curvature in spacetime, which then causes a gravitational pull. Further, gravitational information travels at the speed of light. Even with all these beautiful results, a great problem still exists in physics today: the problem of scale. Gravity works in the macro world, with masses of suns, planets, and solar systems, but not in the quantum world of electrons and quarks. The quest of scientists today is to find a grand unified theory that will bring the 2 worlds together. ■

What did Riemann say that was revolutionary and stunned even the great Gauss? The key notion is this—Riemann separated the topology of the space from the geometry on that space.

Names to Know

Riemann, Bernhard (1826–1866): A student of the great Gauss, Riemann revolutionized the study of shapes by separating topology and geometry into two worlds with the brilliant notion of a metric.

Thurston, William (1946–): A pioneer in the field of topology in dimension 3, Thurston gave us the geometrization conjecture, describing the possible geometries of all 3-manifolds. He was awarded the Fields Medal in 1982.

Suggested Reading

O'Shea, *The Poincaré Conjecture*.

Questions to Consider

1. Find a piece of fruit or a vegetable and pick any two points on it. Find the geodesic between these two points.

2. Look at Jeffrey Weeks's *Curved Spaces* program again. As you fly around different 3-manifolds, try to guess the type of geometry they have simply by how the space in the world feels.

Visualizing in Higher Dimensions
Lecture 29

I want to encourage you to see dimensions in a new way. Explore the world around you, as well as the graphics you see in newspapers and articles, to see how dimension is depicted.

With Einstein, we saw that our 3-dimensional world cannot be separated from time, giving us a 4-dimensional world of spacetime. We were, however, left with a dilemma: How can the micro world and the macro world be explained when brought together? String theory offers a possible solution, using the assumption that underlying the elementary particles we're familiar with are structures called strings. Different resonating patterns of the strings are thought to correspond to different particles. Theoretically, string theory makes gravity and quantum mechanics work together.

String theory has a remarkable consequence, that is, that the universe is not 4-dimensional but 10-dimensional, made up of 9 dimensions of space and one dimension of time. The hidden 6 dimensions are so small that we cannot see them, but we inhabit them. They are called Calabi-Yau manifolds. The dimension of a shape is just the number of pieces of information needed to pinpoint a location on that shape. How many dimensions does it take to describe the world around us? A natural answer is 3, given that we have 3 degrees of movement. But those are simply the spatial directions. To describe the world around us, we need far more than 3 dimensions. Time, temperature, barometric pressure, intensity of light—all are different concepts of dimension. Charles Minard's depiction of Napoleon's invasion of Russia helps us to visualize hidden dimensions in a more tangible way.

We can use color as a variable of freedom to try to understand what it feels like to live in a higher-dimensional world. Consider our 3-dimensional world with 2 of the dimensions as a plane and the third dimension, height, as a spectrum of color, moving from black to light. In this world, a 2-dimensional creature with 3-dimensional powers could escape from a 2-dimensional prison simply by changing colors. Similarly, a 4-dimensional prisoner could escape a 3-dimensional prison.

We now try to use our understanding of 4-dimensional powers with knots. Knots have been described as 1-dimensional circles in 3-dimensional space. The different ways they are placed in 3-dimensional space yield different knots. In 4-dimensional space, the same 1-dimensional knot would have 4 different ways of moving, and all knots could be made into the unknot by isotopy. Note that in both 2 dimensions and 4 dimensions, we get the unknot, but in 3 dimensions, we get structure, which enables us to have such amazing things as knotted DNA, molecular entanglement, quantum cryptography, and string theory.

What is most amazing, however, is that string theory theoretically makes gravity and quantum mechanics work together.

A 2-dimensional jump is key here. To study knots in 4 dimensions, we must take a 2-dimensional surface and place it in 4 dimensions. Placing a sphere in 4 dimensions yields images that are far from intuitive. This is the world of 2 knots. Here, the Roseman moves are the equivalent versions of Reidemeister moves, but to understand these, we must think of them as "movie" moves. ■

Suggested Reading

Adams, *The Knot Book*.

Tufte, *Envisioning Information*.

Questions to Consider

1. Find a descriptive map (a subway map, a historical map) and count the dimensions captured by it.

2. Draw movies of your favorite surface, captured (sliced) from different viewpoints.

Polyhedra in Higher Dimensions

Lecture 30

For us to understand 4-dimensional polytopes, the 3-dimensional spherical shell is what's exciting—just like for us to understand 3-dimensional polyhedra, the 2-dimensional shell is what's exciting.

This lecture focuses on higher-dimensional versions of polyhedra. Is there something like the platonic solids in 4 dimensions? The answer is yes, and we will learn how to visualize these objects using the power of **Schlegel diagrams**.

Just as a manifold describes a surface in higher dimensions, a **polytope** describes a polyhedron in arbitrarily high dimensions, although we'll focus on just the 4-dimensional versions. These structures provide a stepping stone to understanding 4-manifolds. Polytopes are made up of vertices, edges, faces, and chambers. The polyhedral faces of a polytope are glued together to bound a 4-dimensional volume.

One way to explain a cube to a 2-dimensional creature would be to open up the cube and, topologically, lay it flat. This method yields the Schlegel diagram of the cube, which preserves all combinatorial and topological properties of the cube but destroys geometry.

The simplest 4-dimensional polytope is a higher-dimensional version of the tetrahedron called the 4-simplex. In general, a simplex refers to the generalization of a triangle. A 0-simplex is a point; a 1-simplex is a line; a 2-simplex is a triangle; a 3-simplex is a tetrahedron; and a 4-simplex is the 4-dimensional version of this.

To get an n-simplex, in general, we take the convex hull of the $n - 1$-simplex with another point in a new dimension. To build a 1-dimensional simplex (a line), we take a point, another point in a different dimension than the original point, and the convex hull. We build the next one in the same way. The pattern of the simplices is as follows: The edge has 2 points, or zero simplices is its boundary; the triangle has 3 edges, or one simplex is its boundary; the tetrahedron has 4, and the triangle is its boundary. Thus, the 4-simplex must have 5 tetrahedral chambers as its boundary.

To get the n-dimensional cube, we take the $n - 1$-dimensional cube and push it perpendicularly. Similar to the 4-simplex, there is a pattern for the 4-cube: The vertices of the interval have 2 points on the boundary; the square has 4 intervals on the boundary; the cube has 6 squares on the boundary; thus, the 4-dimensional cube has 8 cubes on its boundary.

As a manifold describes a surface in higher dimensions—giving 3-manifolds, 4-manifolds, and higher—a polytope describes a polyhedron in arbitrarily high dimensions.

There are 6 regular polytopes, which are the natural generalizations of the platonic solids to 4 dimensions: the simplex, the cube, the cross, the 24-cell, the 120-cell, and the 600-cell. We can build other polytopes by multiplication, but we need to multiply objects whose total dimension sums to 4. We see some Schlegel diagrams that show the results of a tetrahedron multiplied by an interval, a triangle multiplied by a triangle, and a triangle multiplied by a square. We also see how to draw a Schlegel diagram using the example of a triangle multiplied by a square. A software package called *Jenn3D* allows us to see what things would look like if we were on the surface of a 3-dimensional sphere flying in a 4-dimensional polytope. ■

Important Terms

polytope: The higher-dimension version of a polygon and a polyhedron.

Schlegel diagram: A diagram of a polytope that allows it to be depicted using one less dimension.

Suggested Reading

Cromwell, *Polyhedra*.

Questions to Consider

1. Using the *Jenn 3D* software, fly around in the world of four-dimensional polytopes. Notice how the vertices, edges, and faces behave here compared to three-dimensional polyhedra.

2. Draw two different Schlegel diagrams of a four-dimensional polytope from two different faces that have been removed.

Particle Motions

Lecture 31

I compare mathematical maturity to cooking a complex dish. All the pieces might be simple and well understood, but putting it together is intricate, delicate, and becomes sophisticated.

In this lecture, we look at higher-dimensional work, which is vital to the study of subatomic particle motions, protein modeling, and robot motion planning. Motion planning is based on 3 questions: Is it possible for the robot to carry out the assigned task? If so, what are all possible ways the task can be performed? Finally, what is the best possible way to perform the task?

A mathematical tool called the configuration space can be used to keep track of all possible motions of a robot. Any configuration space requires elements and a relationship between the elements. On a chess board, the elements are the individual formations of the pieces, and each move corresponds to a relationship between 2 elements of the space, the formation before and the formation after. In this example, the chess board itself is the base space, but the configuration space is a meta-space; it's above the chess board, measuring all possible movements. Consider the movement of subatomic particles in a box. Each particle has 2 dimensions of freedom; it can move in 2 possible directions. Given that we have 3 particles and 2 dimensions for each, the configuration space must be 6-dimensional.

For our purposes, we impose 3 rules on the study of configuration spaces of particles: First, the particles must move along lines. Second, the particles cannot collide, and third, they cannot move past obstacles. The configuration space of one particle on an interval is an interval itself. For 2 particles on an interval, the configuration space is a triangle. For 3 particles on an interval, the configuration space is a tetrahedron. In general, a configuration space of n particles on an interval gives us an n-simplex. In the case where we have independent particle motions broken up by a fixed point, the base space can be arranged in any combination, while the configuration space stays the same.

To capture the underlying topology of the base space, we have to adjust our rules to allow particle collisions. Particle movements are still restricted to lines, but now particles can collide and can move past fixed points. Looking at some examples, we see collision lines that give us the underlying topology of the base space.

Under these rules, the first particle can have an entire space of freedom, as can the second particle. If the space is denoted as S, and if we have n particles, then the configuration space is $S \times S \times S$, a superstructure. Note, too, that the intervals cannot be arranged in any order. If the particles are all lined up on one long interval, then the configuration space is an interval × an interval × an interval, depending on the number of particles. If the arrangement of the intervals where the fixed points are is a circle, then the configuration space will be a circle × a circle × a circle—a torus.

We close by looking at the configuration space of 5 particles in the plane, which will be a 10-dimensional space. What is the fundamental group of this space? To build a fundamental group, we need a base point and loops. The base point is any particular configuration of the 5 particles. The loops turn out to be braids of 5 strands. ■

Suggested Reading

Devadoss and O'Rourke, *Discrete and Computational Geometry*.

Questions to Consider

1. Draw the space in which three particles can move *and* collide on a circle. This will be a 3-torus (a cube with identifications). Make sure to draw in the places of collisions.

2. Convince yourself that adding two loops in the fundamental group of the configuration space of points in the plane is the same operation as adding braids.

Particle Collisions

Lecture 32

This notion of manipulating configuration spaces is a powerful idea and a sophisticated one. It is advanced in terms of levels of abstraction where we are actually looking at ideas at the level of research papers being published today.

In the last lecture, we considered different kinds of base spaces; in particular, we looked at particles moving along intervals, circles, and the plane. In this lecture, instead of changing the base space, we manipulate the configuration space, the space of possibilities itself. Several important ideas are related to this field, notably, topological quantum string theories and the world of particle collisions.

Instead of focusing on particle motions, we focus on particle collisions. In this world, particles can go past one another and past fixed particles. These collisions appeared as lines in our configuration space. We are now interested in keeping track of all possible ways collisions could occur.

In the language of mathematics, the different ways of keeping track of collisions correspond to different "compactifications" of the configuration space. The word "compactify" corresponds in a configuration space setting, in terms of particle motions, to keeping track of collisions. There are 2 main types of compactifications that are of interest to us: the naïve compactification and the Fulton-MacPherson compactification.

The naïve compactification is what we looked at in the previous lecture; it's about the collision itself. Two particles colliding on an interval corresponds to a triangle, the 2-dimensional simplex. We can draw this simplex in detail and label all the edges and vertices. The Fulton-MacPherson compactification goes deeper, asking what happens in terms of the collisions themselves.

One way we can extract information about how particles have collided is to reach into the configuration space and truncate it. This truncation converts the point—the point that was tracking the collision of 2 particles and a fixed particle—into an interval of possibilities. If we zoom into the interval, we find that we have a mini-configuration space. What used to be a point has been replaced with an edge, an edge of possibilities, keeping track of this configuration space. Truncating the edge converts the line into a face, extracting more information.

On a side note, [mathematician] Jim Stasheff once remarked to me—he said, triangles, squares, and hexagons are common objects in math, but a pentagon—if that appears in your research—it's something to take notice. There's some associativity going on in disguise.

The associahedron is the polytope that keeps track of the Fulton-MacPherson compactification of particles on lines. This structure is made up of 3 squares and 6 pentagons. In studying this polytope, we enter the world of current mathematics research. We can build this object algebraically by looking at the property of associativity and the ways it can fail. We can also build it from a simplex, as we have seen, and from a convex hull.

We've seen that if we truncate special collections of faces of the simplex, we get the associahedron. What if we truncate all the faces of the simplex? The polytope that results from this maximal set of truncations is called the permutahedron. We have seen this structure before in the Kelvin cell, which Lord Kelvin claimed to be the best way to partition 3-space into equal volume. ■

Suggested Reading

Devadoss and O'Rourke, *Discrete and Computational Geometry*.

Questions to Consider

1. Draw and label the vertices of the three-dimensional associahedron.

2. Can you count the number of vertices for the four-dimensional associahedron? Can you find a general formula to count the number of vertices for the n-dimensional associahedron?

Evolutionary Trees

Lecture 33

A phylogenetic tree is a mathematical tree structure which shows the relationship between species believed to have a common ancestor.

In this lecture, we use configuration spaces to explore the world of evolutionary change. We know that living things inherit traits from their parents. In the mid-1800s, Gregor Mendel showed that these traits are discrete combinations of genes, rather than a continuous blend. The big question in evolutionary theory today is to find the correct relationship between certain organisms based on genetic data. This is the field of phylogenetics.

One way to keep track of relationships between organisms is a phylogenetic tree, which is a mathematical tree structure that shows the relationship between species believed to have a common ancestor. We see, for example, a phylogenetic tree of cats. Such structures are also useful for studying languages, cultures, even literature.

Because different genes give different tree structures to compare, we need to consider not just one tree but an entire space of trees. We then average these trees to get another tree that answers our original question. Each tree has a root—the foundational place where the tree comes from. Each tree has branches; from a mathematical perspective, these are the edges. Trees also have leaves; these are the vertices that have only one edge incident to them. The internal edges and nodes are crucial for communicating relationships.

How do we go about building a space of trees? Our tree space will be a configuration space in which each point in the configuration space is a tree. To build this space that keeps track of trees, we must look at geometry, as well as topology. Each point in our tree space is a tree where the internal branches have lengths, and the lengths somehow encode evolutionary time. If we have a branching of 1, then a branching of 2 and 3, the length of the edge that pulls away from 1 represents time. The length can shrink down to

0, or we can start pulling the 2 and 3 away, and as we do, we're saying that time has evolved in terms of when this partition took place.

In a space of trees with 3 leaves, we have 4 different tree structures. In this space, each tree with an internal edge is represented by a ray, giving us a kind of trivalent branching. In a space of trees with 4 leaves, we get 15 squares (quadrants of the plane) in a pentagonal formation. In general, for n leaves, we get an n – 2-dimensional configuration space.

There are numerous models that can be used to construct spaces of trees. We look at one that uses compactification and the associahedron. This space shows up in work related to both phylogenetics and theoretical physics, and a special version of it is foundational to ideas in string theory and algebraic geometry. ■

Questions to Consider

1. Do some light research to find tree structures that keep track of other relationships, such as languages, ethnicity, or cultures.

2. Try to redraw part of the phylogenetic tree space tiled by 15 squares. As you move around this space, notice how the shape of the tree (which is given by each point in the space) changes.

Chaos and Fractals

Lecture 34

If you just move a little bit on either side of that interval, things go to infinity. A simple flap of a butterfly's wings is enough to push you over, and you get into a chaotic behavior. It doesn't smoothly transition from nice to chaotic; it goes intensely from one point being finite … to one touch over and you're blown away and you just go to infinity.

Thus far in our lectures, we have looked at the world in classical dimensions. Are there objects in nature, such as snowflakes, crystals, or lightning, that do not fit into this framework? Each of these objects has the characteristic of self-similarity; that is, part of its shape looks like the whole. Our work here will bring in the idea of chaos, defined as the behavior of certain systems that are highly sensitive to initial data. As a meteorologist in the 1960s, Edward Lorenz noticed that simple differential equations could have very sensitive dependence to initial conditions. This is called the butterfly effect.

What does chaos mean from a mathematical viewpoint? Consider a simple model of a weather system. We have a function to predict tomorrow's temperature from our current temperature: $f(x) = x + 2$. The orbit of a point under a function is what happens to it as we continue to iterate the function. For example, the orbit of point 1 under our function is that 1 becomes 3, then 5, then 7, and so on.

The most important type of orbit is a fixed point, one that never changes under iteration, but our function doesn't have this feature. Other functions, such as $y = x^2$ or $f(x) = x^2$, have these fixed points. Geometrically, we can graph functions to further understand iterations and orbits. Another function, $f(x) = x^2 - 3$, sends us into the world of chaos. With this function, it appears that the orbits of all the starting points on the graph, the initial values, go to infinity. We notice, however, that the endpoints of any given interval, I_1, are fixed, as are the endpoints of any interval that gets sent to I_1. As we continue to iterate the function, we see that all the points on the intervals go to infinity, but the boundaries of these intervals remain finite. This is chaotic behavior.

To find out which points stay in a finite realm rather than go to infinity requires us to construct a Cantor set. We start with an interval, 0 to 1. We then iterate the process of removing the middle third of that interval to get twice as many intervals as we had originally. If we do this forever, the result is the Cantor set. This is exactly the procedure that's going on with the function $f(x) = x^2 - 3$. The set of elements that do not go to infinity for this function looks exactly like the Cantor set. Further, using analysis, we can prove that after all the iterations, the Cantor set contains no intervals at all. The result is an uncountable number of disjoint points. The Cantor set also exhibits a high level of self-similarity.

It is not only in weather patterns that chaotic behavior is observed. It has been noted … in mathematical biology, such as in population dynamics, and even in the beating of the human heart. Even the motion of Pluto, the once former planet, seems to exhibit this kind of chaotic behavior.

The Sierpinski triangle is a famous **fractal**. Similar to the Cantor set, this fractal can be built from infinite removals. Another well-known fractal is Koch's snowflake, which can be built from infinite additions. This fractal has an amazing property: It has finite area but an infinite perimeter.

We can measure the level of self-similarity in fractals using a new concept of dimension that is based on the number of copies of self-similar sets in the fractal and the magnification level of each such set. Thus, the fractal dimension of a set is defined as the logarithm of the copy divided by the logarithm of the magnification. For the Sierpinski triangle, this calculation yields a dimension of 1.584; for Koch's snowflake, 1.261. ■

Important Term

fractal: An object that holds a high level of self-similarity.

Suggested Reading

Devaney, *A First Course in Chaotic Dynamical Systems*.

Questions to Consider

1. Draw your own fractal shape and try to compute its dimension.

2. Construct (or describe) an object whose fractal dimension lies between 3 and 4.

Reclaiming Leonardo da Vinci
Lecture 35

If I have taught you anything over these lectures, it is that mathematics is not about numbers or equations but ideas themselves. Indeed, I believe that we can reclaim the time of Leonardo once again, that these 2 worlds can once again meet in a powerful way.

Throughout these lectures we have seen the interaction of mathematics and the sciences, but in this lecture, we'll look at the blending of math and visual art. The Renaissance figure Leonardo da Vinci was an unparalleled genius at bringing together these 2 worlds—artistic vision and scientific study. Today, although these worlds have grown apart, I believe that mathematics and art can have an extremely valuable dialogue.

My research interests are in visualizing mathematics, and I have learned that I can actually use my own scribbled pictures to prove new results, to push the frontiers of mathematical research. If I can do that with pictures I've drawn, how much more could someone do who has been trained in the visual arts?

We see some of the classical interactions of mathematics and art in such works as the *Flagellation of Christ* by Piero della Francesca (perspective), Marcel Duchamp's *Nude Descending a Staircase* (an expression of 4 dimensions—spacetime), and M. C. Escher's *Circle Limit IV* (hyperbolic geometry). None of these works, however, pushes the frontiers of mathematics.

One artist who has taken the intersection of mathematics and art to a new level is Sol LeWitt. In his *Serial Project 1*, the artist seems to be exploring the world of configuration spaces. Most of LeWitt's work consists of a set of "instructions" that he gives to another artist, and it's that artist's job to follow the instructions to get a beautiful structure. Another artist who engages in the struggle with shape and design is the sculptor Anish Kapoor. His *Cloud Gate* at the Millennium Park in Chicago uses the idea of curvature and geometry to capture the city's landscape. Joshua Davis, a graphic artist, uses *Adobe Illustrator* and his own software program to create works of art that explore thousands of combinations that can be created with shape, pattern, and color.

Finally, the work of artist **Julie Mehretu** is based on the layering of ideas and images to depict motion, change, space, structure, and history.

In our lecture on phylogenetics, we saw that the movement of 3 particles in different ways gave us the associahedron under the Fulton-MacPherson compactification. This is a result I proved essentially using pictures. I believe this type of visualization can actually lead to new mathematical work.

… pictures are a notation, just like equations. The way one uses color and draws the thickness of lines conveys information. It is, in fact, a language similar to equations and numbers.

We've seen the intersection of math and art in origami. For example, NASA is set to launch the James Webb space telescope in 2014, which is designed to fold up and be placed into a rocket, then unfolded when in space. Curved origami may have an even greater impact on science and technology. Another intersection of math and art that pushes the forefront of both worlds is cartography. Cartograms, for example, focus not on area but on different kinds of information, such as population density.

To push these boundaries even further, I believe we need mathematics laboratories in our schools, places to experiment with tangible ideas. This will help return us to the Renaissance and Leonardo, to the collaboration of art and mathematics. ■

Name to Know

Mehretu, Julie (1970–): An artist who focuses on large-scale paintings depicting motion, space, movement, and high-level historical references, Mehretu constructs her work with layers of acrylic paint on canvas. She became a MacArthur Fellow in 2005 and has had exhibitions all over the world, including at the Museum of Modern Art and the Williams College Museum of Art.

Suggested Reading

Lang, *Origami Design Secrets*.

Tufte, *Envisioning Information*.

Questions to Consider

1. Go to an art museum. Are there pieces that you think deal with the intersection of mathematics and the arts in a genuine manner, without insulting either field?

2. Consider a subway map. Do you believe this is art? Why or why not?

Pushing the Forefront

Lecture 36

We must understand the tools and the weapons we need to deal with. These do not have to be complicated. Mathematics is not about building the most complicated and sophisticated system. Even the simplest idea can be used in a powerful and radical way when used correctly.

Our goal for these lectures has been threefold—to show the relationship between nature and mathematics in the realm of shape, to introduce a language to study shape, and to bring us to the forefront of mathematics research. This last lecture offers a backward look at what we have done from a larger perspective.

We have struggled with issues in our world—surfaces and bubbles, mountain terrains, facial recognition software, convex hulls, robot motion planning, configuration spaces, snowflakes, and fractals. These are all things that we can understand and relate to. We've also looked at things in the micro world—knotted molecules, stereoisomers, fullerenes, mutations, DNA, string theory, and particle motions—as well as the macro world—the shape of the universe, curvatures and black holes, and phylogenetic trees.

In solving problems related to shape in mathematics, we need to remember 3 things. First, we need to be conscious of the type of equivalence we want, whether it's congruencies from classical geometry; scissors-congruence; or equivalence under isotopy, homeomorphism, or homotopy. Second, we must understand the tools we have to attack problems, such as coloring, addition or multiplication, or cutting and gluing. Third, we must understand what we want to measure—volume, area, shape, or size. We cannot capture all of the complications and complexities of a shape but only part of it.

Our inability to measure every facet of a shape led us to invariants. Dimension is the simplest invariant and the most fundamental and foundational. We also saw invariants based on numbers, algebraic equations, and groups. At the same time, we encountered some amazing results, ranging from the Jones polynomial to the Poincaré theorem. We explored configuration spaces and phylogenetics, worlds of higher dimensions where algebra, combinatorics, geometry, and topology meet.

Throughout these lectures, several ideas appeared repeatedly in subtle ways. First is the power of visualization. Do not undervalue what can be drawn or modeled. Second is the power of the global. We saw many times that small local changes resulted in global phenomena. Third is the power of construction, which can be far more convincing than some abstract theoretical answer.

These lectures should provide a platform for you to go to the next level. I encourage you to see how shapes around you can be classified, manipulated, and understood using the tools that we have learned. As you begin to work on more problems, you will start thinking about different ways to attack problems, other than just the classical methods.

In closing, we look at a number of open problems in mathematics, some of which we have encountered in these lectures and some of which have been posed by the Defense Advanced Research Projects Agency as challenges for the 21st century. Keep in mind as you continue your study that professional mathematicians are not the only ones who can do new mathematics. Note, too, that every failure offers new knowledge and that perseverance is essential to succeed in any task. I want to encourage you to not stop here with mathematics and its powerful tools to understand shape. I want you not only to continue learning math but doing it and creating it, as well. ■

I believe mathematics comes in 3 stages—you learn it, you do it, and you create it.

Questions to Consider

1. Come up with a new notion of equivalence of shapes. In what ways can this new notion be useful? In what ways is it weak?

2. Spend some time over the next month (at least three minutes a day) trying to solve one unsolved problem posed in these lectures.

Glossary

amphicheiral: An object is amphicheiral if it can be made into its mirror image.

configuration space: The space that keeps track of all possible ways an object (or a collection of objects) can be arranged.

convex: An object is convex if the line segment containing any two points in the object is contained within the object.

convex hull: Given a point cloud, its convex hull is the smallest convex set containing the point cloud.

dimension: An invariant given to a point on a shape that measures the degrees of freedom afforded at that point.

Fields Medal: The highest honor given for mathematical research; recipients must be under 40 years of age.

fractal: An object that holds a high level of self-similarity.

group: An algebraic structure given to a collection of elements with a means of combining the elements (composition) satisfying three conditions (identity, inverse, associativity).

homeomorphic: A notion of equivalence, weaker than isotopic. Two objects are homeomorphic if one object can be cut up into pieces, stretched, and reattached along the cuts to form the other object.

homotopic: A notion of equivalence, weaker than homeomorphic. This notion deals only with continuous deformations where self-intersections are allowed.

isotopic: A notion of equivalence, the strongest in the world of topology. Two objects are isotopic if they differ by stretching (rubber sheet geometry).

knot: A circle placed in three dimensions without self-intersections.

manifold: A generalization of a surface to higher dimensions, where each point on the manifold has a neighborhood having the same dimension.

phylogenetic tree: A mathematical tree structure that shows the relationship between species believed to have a common ancestor.

polytope: The higher-dimension version of a polygon and a polyhedron.

Schlegel diagram: A diagram of a polytope that allows it to be depicted using one less dimension.

scissors-congruent: A notion of equivalence. Two objects are scissors-congruent if one can be cut up and rearranged into the other.

surface: An object on which every point has a neighborhood that has two degrees of freedom.

surgery: The process of cutting and regluing 3-manifolds.

Biographical Notes

Cauchy, Augustin-Louis (1789–1857): A powerhouse in analysis and a prolific writer, who gave us the arm lemma and the rigidity theorem for polyhedra.

Conway, John H. (1937–): Conway is a professor at Princeton and a prolific mathematician whose works encompass geometry, group theory, number theory, and algebra. In particular, he is known for his Conway notation for knots and links.

Dehn, Max (1878–1952): A student of David Hilbert, Dehn is known for his work in geometry and topology, particularly Dehn invariants for scissors-congruence of polyhedra and Dehn surgery for manipulating 3-manifolds.

Demaine, Erik (1981–): Demaine is an expert in computational geometry who became a professor at MIT at the age of 20. He has solved numerous unsolved problems, such as the one-cut theorem and the Carpenter's Rule problem, becoming a MacArthur Fellow in 2003.

Euler, Leonhard (1707–1783): One of the greatest mathematicians of all time, his scientific works cover analysis, number theory, geometry, and physics. He was one of the first to use topology, from which we receive the formula $v - e + f = 2$ of a polyhedron.

Fejes-Tóth, László (1915–2005): One of the fathers of modern discrete geometry, his works influence practically all areas of this field today. In particular, he investigated packings and partitions and laid the framework for understanding the Kepler conjecture, later solved by Hales.

Gauss, Carl Friedrich (1777–1855): Known as the Prince of Mathematics, Gauss is considered by many to be the greatest mathematician since antiquity. His foundational work in all areas of mathematics continues to influence our world today. We get the notion of curvature and the powerful Gauss-Bonnet theorem from him.

Hales, Thomas (1958–): Hale solved two powerful open problems (some of the oldest ones in discrete geometry): the Kepler conjecture and the honeycomb conjecture. He believes that partnership with computers will be fundamental in solving future mathematical problems.

Jones, Vaughan (1952–): Winner of the Fields Medal in 1990, he created one of the most powerful knot invariants.

Kelvin, William Thomson, Lord (1824–1927): A powerful scientist, Kelvin had wonderful notions of shape and nature. He believed that knots embodied properties of atoms and worked with soap bubbles to posit an efficient tiling of space. He is best known for his Kelvin temperature scale of absolute zero.

Kepler, Johannes (1571–1630): A mathematician and astronomer, Kepler tried to relate platonic solids to the solar system. He also made a conjecture about the best way to stack spheres in space. He is most well known for discovering the elliptical motions of planets around the Sun.

Klein, Felix (1849–1925): Klein spearheaded some of the pioneering relationships between algebra and topology. He also showed us how to obtain all surfaces from gluing polygons.

Lang, Robert (1961–): Lang is not only a world-class origami artist, but he is also a leader in the field of mathematical and computational origami, designing and folding previously unimaginable shapes using mathematics.

Mehretu, Julie (1970–): An artist who focuses on large-scale paintings depicting motion, space, movement, and high-level historical references, Mehretu constructs her work with layers of acrylic paint on canvas. She became a MacArthur Fellow in 2005 and has had exhibitions all over the world, including at the Museum of Modern Art and the Williams College Museum of Art.

Perelman, Grigori (1966–): Perelman completed the work of Richard Hamilton, using Ricci curvature flows to prove the Poincaré conjecture and, most likely, Thurston's geometrization conjecture itself. Although he won the Fields Medal in 2006, he did not accept it.

Poincaré, Henri (1854–1912): One of the greatest and most prolific mathematicians in history, Poincaré worked in geometry, algebra, number theory, physics, and the philosophy of science. He is credited with being the father of modern topology.

Riemann, Bernhard (1826–1866): A student of the great Gauss, Riemann revolutionized the study of shapes by separating topology and geometry into two worlds with the brilliant notion of a metric.

Thurston, William (1946–): A pioneer in the field of topology in dimension 3, Thurston gave us the geometrization conjecture, describing the possible geometries of all 3-manifolds. He was awarded the Fields Medal in 1982.

Weeks, Jeffrey (1956–): A student of Bill Thurston's, Weeks uses his understanding of 3-manifolds to study the shape of the universe. He became a MacArthur Fellow in 1999 and wrote the program *Curved Spaces*.

Witten, Edward (1951–): A mathematical powerhouse who received the Fields Medal in 1990, Witten is considered the greatest physicist of our time, known for his work in string theory.

Bibliography

Adams, Colin. *The Knot Book*. Providence, RI: American Mathematical Society, 2004. This is one of the first and simply the best book on knots and links. Written at an elementary level, it provides numerous details on the subject, ranging from elementary notions of knots, to surfaces, to 3-manifolds.

Aste, Tomaso, and Denis Weaire. *The Pursuit of Perfect Packing*. 2nd ed. Boca Raton, FL: Taylor and Francis, 2008. Written in the form of short chapters touching on numerous points in science and nature, this book addresses such issues as the Kepler conjecture, the Kelvin cell, and the Weaire-Phelan structure. It includes beautiful examples from physics, biology, chemistry, and engineering dealing with packing and partitioning.

Cromwell, Peter. *Polyhedra*. New York: Cambridge University Press, 1999. This is probably the most accessible source for the understanding of polyhedra in all aspects. It has beautiful illustrations and covers numerous topics, such as regularity, rigidity, Gauss-Bonnet, and colorings.

Devadoss, Satyan, and Joseph O'Rourke. *Discrete and Computational Geometry*. Princeton, NJ: Princeton University Press, in press. The only book of its kind, this text brings a topic from computer science into the realm of mathematics. It is a beginning college-level book that discusses such ideas as scissors-congruence, triangulations, Voronoi diagrams, and convex hulls from a geometric viewpoint.

Devaney, Robert. *A First Course in Chaotic Dynamical Systems*. Boulder, CO: Westview Press, 1992. An advanced undergraduate book on chaos and dynamics with lots of details; presented in a quite readable fashion.

Hatcher, Allen. *Algebraic Topology*. New York: Cambridge University Press, 2001. This graduate-level book is one of my favorites. It is elegantly written, with clean and clear illustrations (for a graduate-level math book). It shows algebraic topology from a very visual perspective.

Lang, Robert. *Origami Design Secrets*. Natick, MA: AK Peters, 2003. This book is the magnum opus of the world's greatest origami expert. Although not written at an elementary level, it is well organized, beginning with simple techniques and moving toward deep and powerful tools for folding and design. It has the strongest mathematical underpinnings of any origami book and provides designs for some the greatest creations ever produced in this art form.

O'Shea, Donal. *The Poincaré Conjecture*. New York: Walker and Company, 2007. A mathematical sweep of history is seen through the lens of the Poincaré conjecture in this book, written for non-experts. It tracks the geometric and topological struggles and (most importantly) failures of this great problem, bringing us to the great solution by Hamilton and Perelman.

Richeson, David. *Euler's Gem*. Princeton, NJ: Princeton University Press, 2008. In this book, we see the world of topology and geometry through the eyes of Euler's formula and its generalizations. It's a fun read that shows the power of mathematics weaved into a historical framework.

Tufte, Edward. *Envisioning Information*. Cheshire, CT: Graphics Press, 1990. In one of my all-time favorite books, Tufte helps us to see beautiful and efficient ways of displaying data. This book is accessible and stunning to behold; it serves as a launching point in struggling with the visualization and depiction of shapes.

Weeks, Jeffrey. *The Shape of Space*. 2nd ed. Boca Raton, FL: CRC Press, 2001. A beautifully written, easy-to-read book on building and understanding 3-manifolds. It deals with the geometry and the topology of the universe in clear and knowledgeable terms.

Wilson, Robin. *Four Colors Suffice*. Princeton, NJ: Princeton University Press, 2004. This beautifully written book focuses on the history and mathematics of the four-color theorem. It is written at an elementary level, accessible to all.