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Course Guidebook

Professor Steven Strogatz
Cornell University

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Steven Strogatz, Ph.D.

Jacob Gould Schurman Professor
of Applied Mathematics
Professor of Theoretical and Applied Mechanics
Cornell University

Steven Strogatz is the Jacob Gould Schurman Professor of Applied Mathematics and Professor of Theoretical and Applied Mechanics at Cornell University. After receiving his B.A. in Mathematics summa cum laude from Princeton University in 1980, Professor Strogatz won a Marshall Scholarship to Trinity College, Cambridge. He did his doctoral work in Applied Mathematics at Harvard University, where he received his Ph.D. in 1986; afterward, he was awarded a National Science Foundation Postdoctoral Fellowship. In 1989, he joined the faculty of the Department of Mathematics at MIT, where his research was nationally recognized with a Presidential Young Investigator Award from the White House in 1990. Professor Strogatz began teaching at Cornell University in 1994. In 2004, he was appointed as an external faculty member of the Santa Fe Institute.

Professor Strogatz has done seminal research in chaos and complexity theory with applications to physics and biology. A review in *Nature* described him as one of the "most creative biomathematicians of the past few decades." A theme in his work is the hidden mathematics of everyday life; he and his students have explored the geometry of DNA, the rhythms of human sleep, the synchronous flashing of fireflies, the decline and death of languages, and the mathematics behind "six degrees of separation" in social networks. In particular, his 1998 *Nature* paper (with his former student Duncan Watts) titled "Collective dynamics of 'small-world' networks" is the most highly cited article about networks in the past decade across all scientific disciplines. Professor Strogatz's research has been featured in many mass media outlets, including *The New York Times*, *Newsweek*, CBS News, *U.S. News and World Report*, *The New Yorker*, *Discover*, *Nature*, *Science*, *Scientific American*, *Die Zeit*, and *The Daily Telegraph*.

Professor Strogatz has been lauded for his exceptional abilities as a teacher and communicator. In 1991, he was honored with the E. M. Baker Memorial Award for Excellence in Undergraduate Teaching, MIT's only institute-wide teaching prize selected and awarded solely by students. He also has won several teaching awards from Cornell University's College of Engineering, including the 2006 Tau Beta Pi Excellence in Teaching Award, given to a faculty member selected by engineering students for exemplary teaching. At the national level, Professor Strogatz received the 2007 Communications Award—a lifetime achievement award for the communication of mathematics to the general public—from the Joint Policy Board for Mathematics, which represents the four major American mathematical societies. He has delivered dozens of public lectures nationwide and has made many appearances on radio and television, including National Public Radio, BBC, Discovery Channel, and C-SPAN.

His books include *Nonlinear Dynamics and Chaos* (Perseus, 1994)—the most widely used textbook on chaos theory—and *Sync: The Emerging Science of Spontaneous Order* (Hyperion, 2003), which is aimed at nonscientists and was chosen as a Best Book of 2003 by *Discover*. ■

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Above all, I thank my wife, Carole, and our two young daughters, Leah and Joanna. I love you all so much, and I especially appreciated your patience and support during all those months when I was distracted by that chaotic guest in our house. ■

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Scope:

Newton's laws describe the motion of nearly everything under the Sun and give physics great power in its ability to relate cause and effect. Yet the path of a ball in a pinball machine seems to defy human control—it seems chaotic. What is the science of chaos? How can something be chaotic yet follow deterministic laws? This course investigates how apparent disorder arises from extreme sensitivity to initial conditions.

One of the goals of the course is to explain the main ideas of chaos clearly and honestly, in a way that any interested layperson can understand. The material is inherently accessible because it is rooted in our daily experience; the phenomena we will be discussing are familiar, not esoteric. Furthermore, the subject is marvelously broad and interdisciplinary, intersecting nearly every field of human knowledge and endeavor from astronomy and zoology to the arts, the humanities, and business. No doubt that sounds far-fetched, but it really is not because chaos is the science of how things change—and everything changes. The systems we will consider are the dynamical systems all around us, from the variability of the weather and animal populations to earthquakes, Internet traffic, and the wobbles of London's Millennium Bridge. Chaos also sheds new light on the dynamical systems inside us; think of the spontaneous pulsing of our hearts or the electrical firing of our brain cells.

Another crucial feature of chaos adds to its accessibility: No advanced math is needed to grasp its key concepts. Instead, pictures turn out to be more powerful than symbols and formulas. Although we will discuss many mathematical ideas, we will always visualize them and will not manipulate x's and y's. On the few occasions when it helps to resort to algebra, the math will be explained in both words and pictures.

After an introductory lecture explaining what makes the science of chaos so revolutionary, the course begins with the story of how chaos was glimpsed and then lost for nearly a century before reemerging explosively in the 1970s and 1980s. This is a tale of twists and turns—in many ways like a detective story. We devote the first half of the course to it because the intellectual ride is thrilling and fast paced, with fascinating ideas coming from every direction. We will meet Isaac Newton, the champion of order who gave us a picture of the universe so orderly that it seemed to outlaw chaos once and for all. Two hundred years later, a chink in Newton's armor appeared and gave humanity its first glimpse of mathematical chaos. When it was revealed to Henri Poincaré, however, he recoiled from it; the picture he uncovered was horrifying in both its implications and its appearance. We will see why Poincaré's discovery was overshadowed and forgotten for 70 years before being rediscovered in even starker form by Ed Lorenz, a meteorologist at MIT. His work gave us the concept of the butterfly effect, which incredibly had no impact until the 1970s and 1980s, when the tidal wave of chaos theory finally hit.

Along the way, we will learn the core ideas of chaos. By developing these ideas in their historical context, we can feel like we are sitting at the elbows of the masters, following them on their false starts and sudden flashes of insight. Not only is this an enjoyable way to learn the fundamentals of the subject, it also helps us appreciate just what an intellectual feat it was to create the science of chaos in the first place.

The third quarter of the course focuses on fractals: infinitely intricate shapes in which the smallest parts resemble the whole. We will learn why fractals so often appear hand in hand with chaos and what distinguishes them from ordinary, Euclidean shapes. We then will examine their utility in describing a wide variety of erratic phenomena and processes, ranging from the shapes of coastlines and the gyrations of the stock market to the hidden structure of Jackson Pollock's drip paintings.

The final section of the course takes us to the frontiers of chaos research today. We will see how chaos is being used for practical purposes in encryption and space mission design. Then we will look at the medical implications of chaos and the tantalizing but mysterious linkages among chaos, quantum mechanics, and number theory.

The course concludes with a vision for where the science of chaos is headed next, a path that is clear but daunting. Chaos theorists were among the earliest scientists to focus on nonlinear systems in which the whole is more (or less) than the sum of the parts. Many of the major unsolved problems of science today have precisely this character. From cancer to consciousness, these sorts of problems involve thickets of interlocking feedback loops in which everything affects everything else and the simple logic of cause and effect breaks down. Solving such problems will require new ways of thinking, and these problems are going to challenge scientists for decades to come. ■

The Chaos Revolution

Lecture 1

There's a genuine reason for the excitement about chaos theory. People sense, correctly, that there is something revolutionary happening in science. We're starting to find our way through a no-man's-land, a territory that science had long dreaded and feared to explore. This is the realm of chaos—the erratic side of nature, the discontinuous, the jagged, the irregular and unpredictable.

Welcome to chaos! I realize that sounds silly, and it's meant to. I want to approach this course in a playful spirit, to help you enjoy learning about the amazing subject of chaos theory. By now, everyone has heard of chaos theory. It's one of those rare branches of science to have become a pop sensation. James Gleick's bestselling book *Chaos: Making a New Science* (Viking, 1987) was translated into more than 20 languages and sold more than half a million copies. Meanwhile, multicolored images of fractals seemed to be appearing everywhere in the late 1980s—on T-shirts, posters, and screensavers. The superstar of fractals was the Mandelbrot set, in which dazzling structure reveals itself endlessly as you magnify small parts of the image.

Still, it's remarkable that the public became so fascinated by chaos. Why this curiosity about a seemingly arcane branch of math and physics? Some cynics dismissed the public interest in chaos as no more than fascination with pretty pictures and a catchy name. But I believe there's a genuine reason for the excitement. Science, especially physics, has always been obsessed with order, patterns—the lawful side of nature. But in every field, chaos has always been there, an irritating reminder of what we don't know and don't understand. It was there in ecology, in the fluctuations of wildlife populations. It was there in cardiology, in the arrhythmic quivering of a human heart in the moments before death. And it was there in fluid mechanics, in the turbulent motion of the sea and atmosphere. Scientists regarded all these as vexing puzzles, or worse, as pathologies, monsters.

But starting about 30 years ago, a few scientists began finding strange, unexpected connections between different forms of chaos. Geologists noticed surprising patterns in the frequency of earthquakes. The same patterns appeared in the variability of human heart rates and in bursts of traffic on the Internet. The rules of chaos were turning out to be universal—independent of the stuff behaving chaotically—the same for electronic circuits, lasers, chemical reactions, or nerve cells. It was as if disorder was a thing in itself. It didn't matter what was behaving chaotically; the *process* of becoming chaotic was turning out to be lawful—but the laws were like nothing science had ever seen before.

This is a course about the new science of chaos. By the time we're through, you'll never look at the world the same way again. But what *is* chaos, exactly? It's a paradoxical state, a kind of unpredictable behavior in a system governed by deterministic laws.

We can illustrate it by playing with a double pendulum. The pendulum swings back and forth according to Newton's laws. Indeed, Newton's laws describe the motion of nearly everything under the Sun. For almost 350 years they've given physics great power to relate cause and effect. So you'd think we should be able to predict how this double pendulum will behave, and, until about 30 years ago, that's what everyone thought. It was even a textbook exercise to analyze its motion. I remember solving it in my sophomore physics class.

But under certain conditions (that our professor never asked us to consider), it swings crazily. It seems "chaotic." Worse than that, it's unpredictable. How many times will it whirl over the top before stopping? That's the paradox: How can something be chaotic and unpredictable yet follow deterministic laws?

As unlikely as it seems, the analysis of simple chaotic systems like this one have revealed new principles about how the world works—principles so powerful and far reaching that they have reshaped nearly every branch of science. By the end of the course, you'll appreciate why some have called chaos the third great revolution of 20th-century physics, along with relativity and quantum mechanics.

But is it really so revolutionary? Arguably, yes. Chaos breaks many rules of conventional science. First, it asks unusual questions. Unlike relativity and quantum physics, which originated in the study of phenomena that are extraordinarily tiny, fast, or huge, chaos theory addresses questions at the everyday scale of ordinary life—questions that many scientists thought were already answered, or intractable, or unworthy of study (like the irregular dripping of a leaky faucet). Second, chaos takes the focus off the laws of nature and shifts it to their consequences. Third, it relies on the computer and uses it as a laboratory, not just a number cruncher. Additionally, chaos emphasizes holism, not reductionism (but not fuzzy, soft-headed holism—rather, a holism grounded in rigorous science and mathematics). Chaos is radically interdisciplinary in an era of specialization. Experts in fluid mechanics exchange ideas with cardiologists; physicists work with economists. Also, the picture of the world that chaos offers is topsy-turvy. Simple systems show complex behavior and vice versa. Broader academic understanding of chaos and its implications for a range of fields came suddenly. There were no courses in chaos when I was a graduate student; now every research university offers one. For all these reasons, it seems fair to speak of the "chaos revolution."

The goal of this course is to explain the main ideas of chaos clearly and honestly, in a way that any interested layperson can understand. The material is inherently accessible because it is rooted in our daily experience. Chaos is a science of the curiosities all around us: the puffy shape of clouds, the notorious unpredictability of the weather. Furthermore, no advanced math is needed to grasp the key concepts of the subject. Pictures turn out to be much more powerful here than symbols and formulas. So although we'll be discussing many mathematical ideas, we'll always be visualizing them, not manipulating x's and y's. When we do use algebra, it'll always be explained with words and pictures too.

Above all, chaos theory is marvelously broad. Its study intersects nearly every field of human knowledge and endeavor, from astronomy to zoology, to the arts, humanities, and finance. No doubt that sounds far-fetched, but it really isn't, because chaos theory is the science of how things change—and everything changes.

There are at least two good ways to approach the study of chaos theory. We could focus on the purely mathematical side of chaos, which is very beautiful and deep. But my own preference is to take a much broader approach, because I think that best captures the true spirit of the field. So we're going to look at chaos from every conceivable angle—from the standpoint of science and mathematics, of course, but also from historical, philosophical, artistic, and practical perspectives.

[The] study [of chaos theory] intersects nearly every field of human knowledge and endeavor. … No doubt that sounds far-fetched, but it really isn't, because chaos theory is the science of how things change—and everything changes.

The course begins with the story of how chaos was glimpsed in the late 1800s and then lost again for nearly a century before reemerging explosively in the 1970s and '80s. This is a tale of twists and turns, in many ways like a detective story. It's a story of blunders and breakthroughs, rejected papers and serendipitous discoveries—a very human story. We'll devote the first half of the course to it, because the intellectual ride is thrilling and fast paced, with fascinating ideas coming from every direction. We'll meet Isaac Newton, the champion of order, who gave us a picture of the universe so orderly that it seemed to outlaw chaos once and for all. Two hundred years later, a chink in Newton's armor appeared, and that gave humanity its first glimpse of mathematical chaos. But when it was revealed to Henri Poincaré, he recoiled from it. The picture he uncovered was horrifying, in its implications as well as its appearance. We'll see why Poincaré's discovery was overshadowed and forgotten for 70 years before being rediscovered in even starker form by Ed Lorenz, a meteorologist at MIT, in work that gave us the concept of the butterfly effect. And yet, incredibly, it too had no impact until the 1970s and '80s, when the tidal wave of chaos theory finally hit.

By developing the core ideas of chaos in historical context, we can feel like we're sitting at the elbows of the masters, following them on their false starts and sudden flashes of insight. This first part of our journey will take us from planetary astronomy to weather prediction to population biology to condensed-matter physics—a path almost as chaotic as the problems we're studying! And with the help of computer animations and some beautiful pictures and ideas, we'll make the mathematical foundations of chaos theory come to life.

NASA/GSFC/LaRC/JPL, MISR Team.

Scientists use fractal dimensions to characterize the roughness of highly irregular objects, such as coastlines.

The third quarter of the course focuses on fractals—shapes or processes whose structure repeats ad infinitum, such that their tiniest parts resemble the original whole. We'll learn why fractals are so intimately related to chaos—you could think of them as the footprints of chaos, the remnants that chaos leaves behind. And we'll see what distinguishes fractals from ordinary, Euclidean shapes. Then we examine their utility in describing a wide variety of erratic phenomena and processes, such as the jagged coastline of Norway, the gyrations of the stock market, and the hidden structure of Jackson Pollock's drip paintings.

The final section of the course takes us to the frontiers of chaos research today. We'll see how chaos is being used for practical purposes, in encryption and in space mission design. Next we'll look at the medical implications of chaos for epilepsy and cardiac arrhythmias. And then we'll discuss the tantalizing but mysterious linkages among chaos, quantum mechanics, and number theory.

The course concludes with a vision for where the science of chaos is headed next. We'll see why the new ways of thinking developed by chaos theorists may hold the key to unlocking the greatest problems facing science today, from cancer to the mystery of consciousness. ■

Essential Reading

Gleick, *Chaos*, prologue.

Schroeder, *Fractals, Chaos, Power Laws*, chap. 1.

Stewart, *Does God Play Dice?* prologue.

Supplementary Reading

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 1.

Questions to Consider

1. What are the characteristics of a genuine scientific revolution, as opposed to an advance merely hyped as such? What do you think were the greatest revolutions in the history of science?

2. Is "chaos" just another word for "chance" or "randomness"? What exactly do we mean by "chance"?

The Clockwork Universe

Lecture 2

To appreciate … what is so revolutionary about [chaos theory], it will help if we can understand more about what it overturned. … Our main goal is to trace the rise of Isaac Newton's clockwork universe. … But before [that], I think it's worth going back even farther to the origin of the idea of chaos itself. It's one of humanity's oldest and deepest philosophical ideas.

We begin by surveying the diverse concepts of chaos in the creation myths of the ancients. From the dawn of humanity, people have struggled to make sense of the world. Much of life is predictable: sunrise, sunset, the changing of the seasons. But much of life is also unpredictable: plagues and famines, floods and wars.

The ancient Greeks summarized the tension between order and disorder with two opposing words: cosmos and chaos. *Cosmos* means order. (When you use cosmetics, you're putting your face in order.) *Chaos* initially meant the chasm, the abyss, the bottomless pit. Later, it came to mean the primeval state before creation—a state of utter disorder. This sense of "chaos" as utter confusion persisted into the modern era and gave rise to the scientific term "gas" (denoting trillions of molecules in frenzied motion).

The ancient Hebrew word for chaos is *tohu va-vohu*. A rare phrase, it appears in Genesis 1:2, where it describes the primal, chaotic state of the universe before God brings order to it. Notice that according to this tradition, divine intent was required to wrest order from chaos.

There is also the suggestion, which becomes explicit in the New Testament, that the primeval chaos was not merely a dark abyss or an empty void; it was an active, malevolent confusion that God needed to overcome to create the universe. For many ancient cultures, that evil side of chaos took the form of a serpent, sea monster, or dragon. In the case of the Babylonians, chaos was personified by the ocean goddess Tiamat, a sea monster who dwells in the dark, watery abyss beneath the Earth.

Against this backdrop of mythology, it was a tremendous breakthrough to conceive of the world as ruled by natural laws that human intelligence could find and comprehend. In Western civilization, this breakthrough came with the Ionian Greeks of the 7th and 6th centuries B.C.E. For example, Thales predicted an eclipse of the Sun in 585 B.C.E. Pythagoras discovered the laws of musical harmony and proved his famous theorem about triangles. And Euclid wrote a textbook on geometry that established a style of rigorous, logical reasoning that served as a model for centuries afterward.

Their rational spirit inspired the great minds whose discoveries in the 1600s launched the Scientific Revolution. Specifically, Galileo discovered the laws of motion on Earth. These include the law of inertia, and regularities in the motion of falling bodies and swinging pendulums. Johannes Kepler deciphered the laws of planetary motion, thus solving an age-old puzzle, and Isaac Newton explained the laws found by Galileo and Kepler, thus unifying the science of heaven and Earth.

Newton's achievement was so stupendous, and so important to the rest of this course, that we need to pause to examine what he accomplished. He used several tools, which he invented. These tools included his three laws of motion and the law of universal gravitation (which played the role of Euclid's axioms), as well as calculus (for deducing how things move, given the forces acting on them).

Newton's achievement was so stupendous, and so important to the rest of this course, that we need to pause to examine what he accomplished.

His explanations had a radically new character. They relied on a calculus-based tool that enabled Newton to predict a system's changing behavior from moment to moment. In effect, Newton had invented a mathematical soothsayer—a device now known as a "differential equation." Armed with his differential equation (his second law of motion) and his law of gravity, he explained and unified the laws that Kepler and Galileo had found before him. Until then, those earlier discoveries were cryptic facts, disconnected and mysterious.

Newton's account of natural phenomena was different from Galileo's and Kepler's. To understand how, we begin by imagining watching a movie of an orbiting planet or a flying arrow. Kepler and Galileo could describe such motions from beginning to end using the empirical regularities they'd discovered. In effect, they could give you the story line for the whole movie. Newton's laws, in contrast, let you calculate the "difference" between successive frames of the movie, to predict how nature changes in the next instant—hence the term "differential equations." He needed to find a way to integrate the differences between infinitely many such frames to yield a continuous movie. Hence, this part of Newton's method is known as "integrating" a differential equation.

With this, Newton felt he had found a secret key to understanding the universe—a secret so precious that he published it only in code. Translated into modern language, Newton's secret is: "It is useful to solve differential equations." Newton gave humanity something brand new and shocking: its first deterministic law of nature. "Deterministic" means that a system's future behavior is predetermined by its current state and the governing laws.

But by banishing disorder, the new worldview became disquieting in its own way, especially in its moral and philosophical implications. Once set into motion by its creator, the universe would run on its own, its fate determined by Newton's laws, with no room for chance or free will. The universe appeared to be a vast impersonal clockwork.

Yet we still see random behavior all around us. How do we reconcile the traditional, intuitive understanding of chaos with the clockwork system of orderly processes that Newton developed? Newton's successors believed that chaos was a mirage. It merely reflected our imperfect human knowledge of the countless components of any real system and the myriad uncontrolled forces acting on them. In the meantime, probability theory was developed as a practical tool for handling systems dominated by such uncertainties.

Today, we view chaos as much more deeply ingrained in the universe. Even systems with perfectly known, perfectly deterministic laws can be unpredictable—and those are precisely the ones we call "chaotic." Thus, Newton's notion of determinism is crucial to the rest of this course, because one of the defining features of a chaotic system is that it obeys deterministic laws. ■

Essential Reading

Peterson, *Newton's Clock*, chaps. 3–5.

Stewart, *Does God Play Dice?* chaps. 1–2.

Supplementary Reading

Kline, *Mathematics in Western Culture*, chaps. 12–15.

Questions to Consider

1. Why did so many ancient cultures believe the world originated in chaos? What other primal states could one imagine?

2. Is free will possible in a deterministic world? In a random world? Or is free will an illusion?

From Clockwork to Chaos
Lecture 3

Why was [the three-body problem] so intractable? One possibility was that the difficulty was simply technical. … There was a more disturbing possibility, which was … that perhaps there was some kind of deep mathematical obstacle … that prevented the problem from being solved, no matter how clever you were. Even a super-Newton couldn't do it. That turned out to be the case.

We ended Lecture 2 with a cliffhanger: How can chaos survive in Newton's orderly universe? In this lecture and the next, we'll see how chaos squirmed out in the late 1800s by turning the logic of Newton's clockwork on its head. Newton's laws don't forbid chaos; they require it. That was a shocking turnabout—unthinkable to nearly all scientists at the time, including many who still didn't grasp its significance decades later. To appreciate this next chapter in the story of chaos, we need to remind ourselves just how unassailable the Newtonian paradigm seemed at the time.

The Newtonian worldview dominated Western thought for 200 years, from the late 1600s to the late 1800s. Newton and his successors helped create modernity. Their magnificent scientific advances laid the foundations for the creation of cars, skyscrapers, airplanes, electrical power, and everything else that we associate with the Industrial Revolution. Also, Newton was a hero to such Enlightenment philosophers as Voltaire and Locke. His successes in science inspired a sense of confidence that everything in the world, even human affairs, could be understood and tamed by rational thought. The Newtonian spirit is even visible in the language of the Declaration of Independence. Jefferson's phrase "We hold these truths to be self-evident" is straight out of Newton and Euclid.

But by the end of the 1800s, three cracks began to appear in the foundations of determinism. One led to Einstein's relativity; another, to quantum mechanics; and the third, to chaos. Relativity came about because of inconsistencies between Newton's laws of motion and the new science of electricity and magnetism. Einstein resolved them by overhauling our intuitive (and Newtonian) notions of space and time. Quantum mechanics grew out of an attempt to understand the strange phenomena seen in experiments on light and atomic radiation, which defied explanation in Newtonian terms.

Chaos, on the other hand, grew directly out of the same classical problems that had given Newton his greatest successes: the motion of the planets in their orbits. How ironic that the very topic that had triumphantly ushered in the clockwork universe would later play a role in its demise!

How ironic that the very topic that had triumphantly ushered in the clockwork universe [the motion of the planets in their orbits] would later play a role in its demise!

So what happened? How did we get from clockwork to chaos? First, let's recall what inspired the idea of the clockwork universe in the first place. In a beautiful, satisfying calculation, Newton derived the laws of planetary motion from two deeper principles: his law of gravity and his second law of motion (his soothsayer differential equation). The key assumption is that only the Sun pulls on the planet. All other gravitational pulls (from other planets, moons, asteroids, etc.) are ignored because they are much smaller in comparison. Thus, the calculation is known as the solution of the "two-body problem" (planet and Sun). The solution took tremendous mathematical wizardry because of the nonlinear character of the inverse square law of gravity, which makes the resulting differential equation tough to solve. Newton found a trick—a mathematical transformation—that converted his problem to a much simpler, linear one that he could solve. For decades afterward, mathematicians tried to find similar tricks to solve other differential equations, and often they could.

The logical next step was to tackle the three-body problem (Sun and two planets; or Sun, planet, and Moon). The task was made more urgent by improved astronomical observations, which revealed that the planets didn't move in perfect ellipses. The deviations were thought to arise from the gravitational effects of other planets, which Newton had neglected.

Over the subsequent decades, many great mathematicians tried to solve the three-body problem, but no one could. Meanwhile, an approximate method known as "perturbation theory" was invented. It produced some spectacular successes, such as predicting the existence of Neptune (then undetected but later observed in 1846, exactly where the mathematicians said it would be).

Given these successes, it was natural to want more—to solve the three-body problem completely, with no approximations. But the efforts of all the best minds kept failing, just as they had for 200 years. Even when the problem was idealized and simplified drastically, still no one could solve it. One simplification assumed the third body was a tiny speck of dust—pulled on by the other two planets, but exerting a negligible effect back on them—but the problem was still impenetrable.

Why was the three-body problem so difficult to solve? One possibility was that the issue was simply technical—after all, Newton needed tremendous ingenuity to solve the two-body problem, and perhaps the three-body problem was much harder still. A more disturbing possibility—one that in fact turned out to be true—was that there was a mathematical obstacle that prevented the problem from being solved. We can see this through a modern computer simulation showing the orbits in the three-body problem, which can be hopelessly complicated.

In fact, the three-body problem contained the seeds of chaos—a phenomenon as yet unimagined. In saying that, we're now using "chaos" in the contemporary scientific sense of unpredictable, random-looking behavior in a system that is nevertheless governed by nonrandom, deterministic laws. In the next lecture we'll see how this new kind of chaos came to light—and why it was just as quickly buried again. It's an odd little story that holds broader lessons about the sociology of science—and the psychology of scientists. ■

Essential Reading

Peterson, *Newton's Clock*, chaps. 4–6.

Stewart, *Does God Play Dice?* chap. 2.

Supplementary Reading

Cohen, *Science and the Founding Fathers*, chap. 2.

Kline, *Mathematics in Western Culture*, chaps. 16–18, 21.

Questions to Consider

1. Pick any mundane piece of technology (such as a toaster, car, computer, or airplane) and explain how it relies on something that Newton discovered. As far as possible, summarize the chain of intermediate discoveries and ideas that connect the object to Newton's work.

2. Besides the Declaration of Independence, where else did Newton's work have an impact outside of science, say in the arts, humanities, philosophy, or politics?

3. Knowing what we know today about relativity, quantum mechanics, and chaos, what are some of the most serious flaws in the Newtonian view of the universe?

Chaos Found and Lost Again

Lecture 4

Jules Henri Poincaré was a French polymath, a fantastic mind. … He made marvelously original contributions across a wide spectrum of math and physics, in some cases inventing whole new fields. … Poincaré is a founder of topology, and his name lives on in popular imagination today through what's called the *Poincaré conjecture*. … [But] we'll be focusing on his discovery of chaos.

In the next few lectures, we'll see that the path to the discovery of chaos reads like a detective story, with missed clues, false leads, and wrong turns. The first clue comes with Henri Poincaré's groundbreaking work on the three-body problem. Unfortunately, as with love, the course of true discovery never did run smooth—Poincaré blunders at first, then corrects himself, and finally is aghast to perceive what we now call "chaos" and "fractals" lurking in the three-body problem. The implication is that a system governed by deterministic laws can nevertheless behave unpredictably; chaos creeps into the clockwork.

Poincaré (1854–1912) was also a lucid expositor and wrote for the general public on topics ranging from the philosophy of science to the psychology of mathematical creativity. He came from an influential family; his cousin Raymond was the president of France during World War I. He was extremely visual and intuitive, and always brimming with ideas—it was said he flitted from problem to problem like a bee to flowers. But he was also extremely nearsighted, physically clumsy, and pathetic at drawing—as a student, he scored a perfect zero on the drawing exam for entry into the École Polytechnique and needed a special dispensation to be admitted.

Poincaré's work on the three-body problem was triggered by an international contest among the world's finest mathematicians. In 1885, King Oscar II of Sweden and Norway offered a prize for the solution, with the award to be given in honor of his 60th birthday in 1889. An ambitious 31-year-old professor, Poincaré burrowed into the problem for the next 3 years and submitted his entry a few weeks before the closing date.

The approach Poincaré took to the three-body problem was stunningly creative. He invented a new way of thinking, using pictures instead of formulas. It provides tremendous insight with surprisingly little effort. His pictorial method has become central to chaos theory. We're going to develop and use it throughout this course.

To illustrate Poincaré's method, consider a swinging pendulum. There are three ways we could study its motion. One option would be to conduct an experiment. Alternatively, we could try to solve Newton's differential equation. But this requires arcane math (elliptic functions) rarely taught even in graduate school. Our third option would be to draw a picture of the differential equation. This was Poincaré's approach.

Poincaré invented a new way of thinking, using pictures instead of formulas. It provides tremendous insight with surprisingly little effort. His pictorial method has become central to chaos theory.

Our picture would be abstract. It would depict the pendulum's motion as an imaginary point flying on a "trajectory" through an imaginary world called "state space." Continuing our earlier metaphor (from Lecture 2) in which a differential equation is like a movie, a state is like a single frame in the movie, and state space is a gigantic collection of all possible frames, all possible scenes. A trajectory takes you from one frame to the next—it's one possible scenario.

More precisely, the "state" of a system is defined as all the information we need to determine how the system will change in the next instant. A pendulum's state consists of two numbers: its current angle and velocity. Hence, its state space has two axes: one for angle and one for velocity.

Poincaré's great idea was that state space helps us imagine what the movie will look like before we see it, thus enabling us to predict the real system's behavior. Here's how it works: Newton's law (his differential equation) defines a vector at each point in state space. The vector tells the state where to go next, like the arrows showing storms moving on a weather map, or like those old dance lessons with diagrams showing you where to put your feet next, step by step. Following the arrows amounts to solving the differential equation. The resulting trajectory reveals how the pendulum moves. Even if we'd never seen a pendulum swinging or whirling over the top, the picture would disclose those possibilities to us. Similarly, Poincaré hoped the method would reveal all the motions of three gravitating bodies.

So what happened when Poincaré unleashed his method on the three-body problem? He didn't quite solve it, but he illuminated the problem as never before. His entry to the contest won the king's prize: a gold medal and 2500 crowns (equivalent to about half his annual salary).

Unfortunately Poincaré had blundered at an important point. He caught his error, but his manuscript was already being printed. Poincaré stopped the presses and paid for the print run so far—3585 crowns, more than his whole prize. Over the next few months, he worked feverishly to fix his mistake. What he discovered next would change science forever.

His new analysis revealed chaos and fractals lurking in the three-body problem. Poincaré was horrified. The picture showed a certain pair of curves looping back on themselves, crisscrossing to form an infinitely fine mesh (a kind of fractal shape). Neighboring boxes in the mesh represented different long-term motions. Hence two initially similar states could have wildly different futures. This meant that a deterministic system could be unpredictable. Chaos had infected the clockwork.

There is one more twist in this story: Having been so brilliantly uncovered, this first glimpse of chaos was just as quickly obscured, overshadowed, and forgotten. Why? Poincaré never drew a picture of what he had in mind. Was it because of his ineptitude as a draftsman? Computers did not exist, so he could not use computer graphics to help other people see what he was imagining. Only a small mathematical priesthood could follow his arguments. Poincaré didn't welcome chaos. It was an obstacle that prevented solution of the three-body problem. Also, the zeitgeist was against him. Physics was about to be swept up in two other revolutions—relativity and quantum mechanics—and had little interest in such old-fashioned questions. For all these reasons, Poincaré's discovery of deterministic chaos had little impact for 70 years. ■

Essential Reading

Peterson, *Newton's Clock*, chap. 7.

Stewart, *Does God Play Dice?* chap. 4.

Supplementary Reading

Diacu and Holmes, *Celestial Encounters*, chap. 1.

Galison, *Einstein's Clocks, Poincaré's Maps*, chap. 2.

Internet Resource

You can play with a simulation of a pendulum and its motion in state space here: http://www.aw-bc.com/ide/idefiles/media/JavaTools/pndulums.html.

Questions to Consider

1. To check if you understand what we mean by the "state" of a system, figure out how many numbers you need to describe the state of the three-body problem, assuming that each of the bodies is a point particle of the same mass. Your choices are 3, 6, 12, 18, or 24. (Hint: What do you need to know about each body, at a given instant, to predict the future of the entire system?)

2. Can you think of other scientific discoveries, like the discovery of chaos, that weren't appreciated until decades or even centuries later? What were the reasons for the slow acceptance in each case?

The Return of Chaos
Lecture 5

Without any conceptual framework for thinking about chaos, either because people were unaware of Poincaré, or unable to understand him, or just didn't think that his work applied in their context, researchers … dismissed the chaos that they were observing directly, or they ignored it, or I think what was really happening in many cases is they simply couldn't see what they were seeing.

For the next 70 years after Poincaré, chaos remained a quiet backwater of science, studied only by a small mathematical priesthood. This was an era when science was becoming increasingly specialized and mathematics was starting to look inward. Poincaré's flame was kept alive by George David Birkhoff (1884–1944), the first great mathematician to be trained entirely in America, and by a handful of mathematicians in Europe and the Soviet Union. Engineers, meanwhile, were extending Poincaré's methods to analyze the nonlinear oscillations of vacuum tubes and other newfangled technologies relevant to the development of radio, telephones, and radar. Occasionally these researchers stumbled across their own hints of chaos. But because they had no conceptual framework for thinking about it, they dismissed it or ignored it.

The calm ended in 1960 with a thunderclap, due, appropriately enough, to a man fascinated by storms and weather prediction. In this lecture we recount Edward Lorenz's discovery and analysis of chaos in a simplified model of weather patterns.

First, a bit about the man himself. Although Lorenz is universally regarded as one of the pioneers of chaos theory, he's very modest and unassuming. I had the privilege of getting to know Professor Lorenz when I was his colleague at MIT in the early 1990s. Lorenz was born in 1917 and grew up in West Hartford, Connecticut. As a boy, he had always liked weather, but he liked math even more. He thought he would go into math after he graduated from Dartmouth in 1938. He went on to get a master's degree in 1940 from Harvard, where he studied dynamical systems with Birkhoff. But then World

War II broke out, and he found himself working for the Army Air Corps as a weather forecaster. After the war, he continued to be interested in forecasting and wanted to help develop the mathematical theory behind it.

Lorenz wanted to see how well certain forecasting methods would do by testing them on a computer surrogate for the weather. So he needed to make artificial weather. This was a novel strategy. Lorenz was using the computer as an arena for experiments, not as a calculating machine. He kept the computer in his office, also very unusual (personal computers were 25 years away).

His first version of artificial weather was too simple. It would either settle down to an unrealistic equilibrium state or repeat in perfect periodic cycles. Lorenz realized that he needed to concoct a system that was both deterministic and non-periodic, meaning that it would never repeat itself exactly. Otherwise his artificial weather would be easily predictable and hence worthless as a test of forecasting methods. After struggling, he came up with a deterministic system that never repeated itself. (The missing ingredient had been that he needed to vary the warming effect of the sun from east to west, not just from north to south.)

While studying his artificial weather, Lorenz happened across the "butterfly effect," the extreme sensitivity of a chaotic system to tiny changes in its initial conditions (named for an imaginary butterfly flapping its wings in Brazil, creating tiny air currents that ultimately trigger a tornado in Texas).

The discovery came while Lorenz was rerunning a simulation that he had wanted to project farther into the future. He stopped the computer midway through its previous simulation, typed in the numbers it had printed out, and set the computer running again. Then he went for coffee and came back an hour later (by which time 2 months of artificial weather had passed).

The results were nothing like what he had seen the first time! At first Lorenz suspected a computer malfunction (which was not unusual). On closer examination, he noticed that the old and new simulations agreed at first, but then began to differ in the third decimal place, and then in the place before that. The discrepancy was doubling in size every 4 days or so. This clue revealed the culprit: The initial conditions had not actually been the same in the two simulations. To save space on his printouts, Lorenz had rounded off the original numbers to three digits, but the computer was using six digits internally. The initially tiny round-off errors (in the fourth decimal place) were growing exponentially until they eventually overwhelmed the solution itself. The butterfly had changed the weather!

The butterfly effect is the signature of chaos. Understanding it will be crucial to all that follows.

What are the broader lessons of Lorenz's work? The butterfly effect is the signature of chaos. Understanding it will be crucial to all that follows. It is also probably even more severe for real weather than for Lorenz's artificial weather. Hence, long-range weather prediction will never be possible. Even if we had a perfect model of the atmosphere, the inevitable errors in our measurements of current weather conditions will grow until they make our forecast look silly.

There's also a lesson here about the discovery process: Lorenz was serendipitous, not lucky. By deliberately searching for something (a stringent test of a forecasting method), he was keenly vigilant, which helped him spot something unexpected and far more interesting (the butterfly effect).

Lorenz's research strategy changed everything. He adopted Poincaré's pictorial approach but strengthened it enormously by using a modern computer to simulate his system and graph the results. The computer became more than a number cruncher. It became a telescope for the mind, a way of imagining the inconceivable. Suddenly scientists could *see* the consequences of the laws of motion (even though they still couldn't write formulas for them). Computers provided intuition. In retrospect, this is why chaos theory had to wait until the 1960s. Without the computer to perform millions of calculations in the blink of an eye, scientists couldn't begin to fathom what their equations were trying to tell them. ■

Essential Reading

Gleick, *Chaos*, 11–31.

Lorenz, *The Essence of Chaos*, chaps. 3 and 4.

Stewart, *Does God Play Dice?* chap. 7.

Strogatz, *Sync*, chap. 7.

Supplementary Reading

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 9.

Internet Resource

If you want to examine the butterfly effect for yourself in a simulation of another of Lorenz's models (to be discussed in Lectures 7 and 8), try this online Java applet: http://www.aw-bc.com/ide/idefiles/media/JavaTools/lrnzdscv.html.

Questions to Consider

1. Do you believe this butterfly joke should be taken seriously? If a butterfly can trigger a tornado, couldn't it just as well prevent one? If so, what is the likely net effect of butterflies on the weather?

2. What other important discoveries have been made by serendipity?

Chaos as Disorder—The Butterfly Effect
Lecture 6

"We collectively wish to apologize for having misled the general educated public by spreading ideas about the determinism of systems satisfying Newton's laws of motion that, after 1960, were proved to be incorrect."
—Sir James Lighthill, former Lucasian Chair of Mathematics at the University of Cambridge, in a landmark 1986 apology on behalf of all scientists

Lorenz's butterfly effect has now entered popular consciousness, but some aspects of it are commonly misunderstood. In this lecture we clarify its scientific meaning and philosophical significance. The term derives from a 1972 lecture by Lorenz entitled "Predictability: Does the Flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas?" Lorenz originally had a seagull in the title, but the session chair at the conference replaced it with the more poetic image of a butterfly.

Since then, the butterfly effect has intrigued authors and filmmakers. It featured prominently in Steven Spielberg's blockbuster movie *Jurassic Park* (1993), based on Michael Crichton's bestselling novel (1990). In the movie, Jeff Goldblum plays chaos theorist Ian Malcolm, dressed in leather and looking like a rock star. He demonstrates the butterfly effect by flirtatiously holding Laura Dern's hand and placing a drop of water on the back of it, then asking her to predict which way it will roll off. The unpredictability of the droplet's trajectory is a harbinger of the unpredictability of Jurassic Park, an amusement park with a supposedly fail-safe population of genetically engineered dinosaurs that can never reproduce. The butterfly effect also plays a central role in Tom Stoppard's play *Arcadia* (1993) and in the movies *Sliding Doors* (1998) and *The Butterfly Effect* (2004).

But didn't everyone already know that little things sometimes make a big difference? In fact, they did. Lorenz himself mentions two literary precursors. The first is a 1941 novel called *Storm*, which Lorenz's sister gave him for Christmas when she learned he was going to study meteorology. In the novel, a meteorologist remembers that his professor had remarked that a man sneezing in China can set off a blizzard in New York. The second is the Ray Bradbury time-travel story “A Sound of Thunder” (1952), in which the death of a prehistoric butterfly changes the outcome of a present-day presidential election. Indeed, the idea of the butterfly effect is at least as old as John Gower’s familiar verse “For Want of a Nail,” published in 1390.

The surprise was that simple systems could also suffer from the butterfly effect. It was always clear that we couldn’t predict wars or the course of our own lives. There are too many complexities, with millions of unaccounted variables obeying unknown laws, or no laws at all. But for simple systems like double pendulums, or the three-body problem, or Lorenz’s model of artificial weather, we know all the variables and the laws determining their behavior. Yet we still can’t predict what they’ll do in the long run.

Poincaré certainly realized this. He expressed it clearly in his 1914 book, *Science and Method*. Still, for nearly all other scientists, the recognition of this kind of unpredictability was late in coming and philosophically hard to swallow. It led Sir James Lighthill (who held the same chair at Cambridge that Newton once occupied) to issue a remarkable collective apology. In a 1986 article called “The Recently Recognized Failure of Predictability in Newtonian Dynamics,” he expressed regret on behalf of all scientists “for having misled the general educated public by spreading ideas about the determinism of systems satisfying Newton’s laws of motion that, after 1960, were proved to be incorrect.”

The butterfly effect simply doesn’t occur in the kinds of systems most scientists had been studying since Newton. It’s not there in systems that relax to equilibrium or … oscillate in regular cycles.

Why were scientists so slow to recognize the butterfly effect? First, the idea of it was philosophically distasteful, and almost threatening to science itself. Science depends on being able to draw a conceptual box around a system of interest, with the confidence that you can neglect tiny effects coming from outside the system. If you had to think about what’s happening on Mars just to study the motion of a pendulum, science (and life itself) would be impossible.

Also, the butterfly effect simply doesn't occur in the kinds of systems most scientists had been studying since Newton. It's not there in systems that relax to equilibrium or that oscillate in regular cycles. In those cases, small disturbances stay small, or if they grow, they do so very slowly. A pair of metronomes illustrates this. If you start them in sync and then disturb one, the difference between them stays roughly the same thereafter and doesn't snowball. The tides, the return of Halley's Comet, the timing of eclipses—all of these are strongly periodic and hence predictable, because tiny disturbances do not mushroom into major forecasting errors.

The butterfly effect only afflicts systems that are both deterministic *and* non-periodic. One example is Lorenz's artificial weather model. Another is the double pendulum toy we've played with previously.

The butterfly effect does not imply that chaotic systems are unpredictable. They in fact are predictable in the short term because of their deterministic character. But they become unpredictable after a certain amount of time, called the *horizon of predictability*. It's the time required for tiny errors to double in size. For a chaotic electrical circuit, the horizon is something like a thousandth of a second. For the double pendulum, it's a few tenths of a second. For the weather, it's unknown but seems to be roughly a week or two, and for the entire solar system, it's about 5 million years (as determined by very careful computer simulations).

It's because the horizon is so long for the solar system that the motions of the planets seem utterly predictable to us today; and on the time scales of a human life, or even of the whole history of astronomy, they *are* predictable. When we calculate planetary motions hundreds of years into the past or the future, our predictions are reliable. But any claims about the positions of the planets 4 billion years ago, at the dawn of life on Earth, would be meaningless.

The bottom line is that you can never predict much longer than the predictability horizon, no matter how good your instruments become. The exponential growth of errors in a chaotic system overwhelms even the most meticulous observations. ■

Essential Reading

Gleick, *Chaos*, 11–31.

Lorenz, *The Essence of Chaos*, chap. 1, app. 1.

Strogatz, *Sync*, chap. 7.

Supplementary Reading

Crichton, *Jurassic Park*.

Lighthill, "The Recently Recognized Failure."

Poincaré, *Science and Method*, chap. IV.

Stoppard, *Arcadia*.

Questions to Consider

1. If you haven't already, read *Jurassic Park* and *Arcadia*. What do you think of Crichton's and Stoppard's accounts of chaos theory? In each case, does the discussion of chaos add to the drama? Are their treatments scientifically accurate? Illuminating?

2. In what ways has the butterfly effect changed the course of your own life?

Picturing Chaos as Order—Strange Attractors

Lecture 7

There's a cumulative order in a chaotic system, a kind of pattern … but … there's a kind of randomness, too. There is the butterfly effect and … tiny differences get amplified as time goes on, exponentially fast, leading to the impossibility of predicting what this system will do far into the future. … It's that confusing interplay of order and randomness that this strange attractor embodies.

In the last three lectures we've been focusing on the unpredictable side of chaos. But chaos is yang as well as yin, order as well as randomness. This lecture describes a way to visualize an amazing kind of order inherent in chaos. The resulting image is known as a *strange attractor*. Just as a circle is the shape of periodicity, a strange attractor is the shape of chaos. It's "strange" because its geometry is infinitely complex, and it's an "attractor" because the system is always drawn toward the behavior it represents. The existence of strange attractors is tremendously encouraging to scientists. By revealing the unexpected order within chaos, strange attractors offer hope that some forms of it might be partially predictable and controllable. So let's see what strange attractors are, how they were discovered, and why they matter.

We pick up with the hero of Lectures 5 and 6, the meteorologist Edward Lorenz of MIT. Having uncovered the butterfly effect in his artificial weather model, Lorenz next sought the mathematical essence of chaos. Could chaos arise in simpler systems? A colleague pointed him toward a model of convection, a rotating pattern of flow that sets in when a fluid, like air or water, is heated from below and cooled from above.

We can demonstrate all the phenomena of interest with a clever contraption—a tabletop waterwheel built for this purpose by Professor Willem Malkus of MIT. Water flows in steadily at the top and turns the wheel, much like the heat of the sun drives convection in the atmosphere. The waterwheel displays different types of behavior, depending on the amount of damping in the wheel. (We can vary the friction by adjusting a brake.) At high damping, the wheel settles into a uniform rotation. At low damping, the wheel rocks back and forth periodically, like a pendulum. In between, when the damping is moderate, the wheel turns chaotically. It rotates one way and then the other, reversing direction erratically and unpredictably.

Hoping to find order in the chaos, Lorenz sought to visualize the motion. The usual picture—a graph of a variable at different times—doesn't help. It just shows a complicated wiggle.

Instead, Lorenz made brilliant use of Poincaré's notion of state space (see Lecture 4). Recall that a "state" of a system is the collection of variables needed to determine the system's future. For Lorenz's convection model (or its waterwheel analogue), three variables determine the motion. Thus, the state space has three dimensions, with one dimension for each variable. That's wonderful, because the human mind can visualize three dimensions! (Lorenz's earlier weather model had 12 variables and so couldn't be visualized in this way.) Using computer simulation, we can graph the behavior of Lorenz's system as it moves from state to state, sailing like a comet on a trajectory through state space.

Let's start simply. Suppose the system settles down to rotating at constant speed. Then the picture in state space shows a trajectory spiraling down to a single point. The trajectory homes in on this point as if it were being attracted to it. Thus, we say that the system has settled onto an "attractor point."

Attractors are important. We'll be discussing them throughout this course. An attractor represents a system's natural, long-term mode of behavior. If you nudge a system off its attractor by disturbing it slightly, it will soon head back to the attractor. For example, if someone startles you by shouting "Boo!" your heart rate may speed up, but then it quickly relaxes back to normal. Your heart, like many other systems in nature, is carefully regulated to keeps its behavior within certain limits. So if an unchanging steady state corresponds to an attracting *point*, a periodic state (like the waterwheel rocking back and forth) corresponds to an attracting *cycle*. These simplest kinds of attractors—points and cycles—have very familiar geometry. Poincaré had mastered them 70 years earlier.

Now Lorenz was trying to visualize the attractor corresponding to a *chaotic* state. Poincaré hadn't done that. Lorenz's attractor has such weird geometry that it deserves the name *strange attractor*!

Let's try a few ways to explain what a strange attractor is. A computer simulation of the Lorenz system shows a trajectory spiraling around on a structure shaped like—prepare yourself for a twist straight from *The Twilight Zone*—a pair of butterfly wings. The trajectory makes a few loops around one wing before darting over to the other wing, jumping back and forth erratically. A 3-D plastic model of the attractor looks like a pair of surfaces that join at a hinge. Lorenz argued the surfaces could not actually join. Doing so would violate determinism. He concluded that what appear to be two surfaces must in fact be an "infinite complex of surfaces."

A 3-D plastic model of the attractor looks like a pair of surfaces that join at a hinge. Lorenz argued the surfaces could not actually join. Doing so would violate determinism.

This mind-boggling idea can be made more vivid by considering a metaphor involving a nightmarish parking garage, with two interconnected towers and infinitely many levels. This metaphor also illustrates four other key aspects of chaos on a strange attractor: the determinism of the motion, its non-periodicity, its confinement to the strange attractor, and its extreme sensitivity to initial conditions (the butterfly effect).

Strange attractors are important because they teach us that chaos is not the absence of order. It is a marvelously subtle state poised between order and randomness, with both aspects intermingled.

In this lecture we've focused on the abstract shape of chaos. Next we'll explore the structure of chaos in time and see that there is beautiful order in that as well. All of this will come back with great effect later in the course, when we'll see how the order in chaos is allowing it to be used for extraordinary applications in medicine, space travel, and communications technology. ■

Essential Reading

Gleick, *Chaos*, 119–53.

Lorenz, *The Essence of Chaos*, 136–46.

Stewart, *Does God Play Dice?* chap. 7.

Strogatz, *Sync*, chap. 7.

Supplementary Reading

Strogatz, *Nonlinear Dynamics and Chaos*, chaps. 9, 12.

Internet Resource

Play with a Java applet of Lorenz's attractor here: http://www.aw-bc.com/ide/idefiles/media/JavaTools/lrnzr320.html.

For a simulation that ties the mathematical model to the physical problem of convection, try this one: http://www.aw-bc.com/ide/idefiles/media/JavaTools/lrnzphsp.html.

Questions to Consider

1. This lecture focused on strange attractors, which represent self-sustaining chaos. But attractors can also represent states of unchanging equilibrium, or periodically repeating cycles. The temperature in a house with a working furnace and thermostat would be an example of a system that has an equilibrium state as an attractor. (The temperature stays constant, and if disturbed, comes back to the desired set point.) Give other examples of systems with attracting equilibrium states.

2. What are some examples of systems with attracting cycles? (These are systems that relax back to some standard cycle after being disturbed.)

Animating Chaos as Order—Iterated Maps

Lecture 8

Why … is the Lorenz map so important? … Because it's so simple. It's astonishing that there's a pattern like this lurking in chaos. … [The iterated map] captures with great fidelity what is going on in the Lorenz system, and it ignores in-between parts. … Iterated maps are a great boiling down of information in the differential equation.

This lecture continues our exploration of order in chaos. We're going to reveal a new kind of order by using the concept of an *iterated map*. To give you some intuition about this idea, let's contrast iterated maps with two other math ideas we've been developing—differential equations and strange attractors. In metaphorical terms, they're like three kinds of photography.

Suppose we want to film a dancer in a discotheque. (The chaotic system is like the dancer.) We could make a movie of the dancer. Here time flows continuously. Or we could make a time-lapse photograph, a blurry compendium of all his movements. Here all times are shown at once. Or we could turn off the lights and flash a strobe light on him, giving his dance a jerky, psychedelic feel. Here time is discrete.

A differential equation is like a movie (or more precisely, the logic behind a movie). It dictates how the system unfolds, instant by instant, as time flows continuously. But sometimes you don't want all this information. Attractors and iterated maps are two ways to boil the dynamics down to something simpler.

A strange attractor is like a time-lapse photograph. In Lecture 7, we used this technique to expose the long-term, cumulative shape of chaos (the strange attractor). It was as if we went out into the backyard and set our camera to take an overnight exposure of the stars, except that instead of seeing the stars moving in perfect circles—the simple kind of order that enthralled the ancient Greeks—we saw something much more modern and mysterious. Our time-lapse picture showed the orbits of the Lorenz system as it moved around in its abstract state space. Instead of forming a tangled mess, as we might have expected from a chaotic system, the orbits traced out a delicate, beautiful shape that resembled a pair of butterfly wings. This was the strange attractor.

Now, in this lecture, we are going to flash a strobe light on chaos, catching it at discrete moments at it winds around the attractor. The goal is to find a rule that animates the chaos from one snapshot to the next. The pattern we'll find (the "iterated map") is the engine of chaos. It drives the chaotic motion that we see in the Lorenz system. In later lectures, we'll see that iterated maps are much more than a quirk of the Lorenz system—they are pivotal to the whole science of chaos, underlying and orchestrating the chaos seen in everything from electronics to animal populations.

We are going to flash a strobe light on chaos, catching it at discrete moments at it winds around the attractor. … The pattern we'll find (the "iterated map") is the engine of chaos.

The iterated map for Lorenz's system comes from looking at a computer simulation of his convection model. We watch the variable that Lorenz called z and track how it changes over time. It moves up and down but doesn't ever repeat exactly (as expected for a chaotic system). Whenever z tops out (reaches a maximum), we record its peak (how high it went). Thus, we observe the system only when it reaches a peak; that's the trigger for our mathematical strobe to flash. Does a peak in one snapshot predict anything about the peak that follows it? To find out, we graph one peak versus the next. As we repeat (or "iterate") this process for all the peaks in the record, the data points hop around haphazardly (because of the chaos they represent), but ultimately a pattern emerges. All the points fall on a curve that looks like a pointy hat, or an upside-down V. It's astonishing that such a simple pattern pops out of the chaos. (If the system were random instead of chaotic, the data would have scattered into a cloud, not a thin curve.) This curve is now called the *Lorenz map*. By iterating this map—using one peak to generate the next—we produce a chaotic string of numbers. This chaotic sequence is the heart and soul of the chaos produced by Lorenz's much more elaborate convection model. The math is no harder than pressing a button on a calculator, over and over again.

There are quite a few larger lessons here, so let's just focus on two for now and save the rest for subsequent lectures. The first lesson deepens an ongoing theme. Specifically, chaos is *not* the opposite of order. It's a mix of order and randomness, in the following senses: Lectures 5 and 6 showed that chaos is unpredictable and random-looking in the long run because of the butterfly effect. But it's predictable and orderly in the short run, because it obeys deterministic laws. Even in the long run, a kind of order persists, as embodied by the strange attractor (see Lecture 7). The system has to stay on the strange attractor, and hence the overall character of its long-term motion is predictable (even if the fine details are not). And now, in this lecture, we've seen that the middle-term dynamics are structured too. Knowing one peak, you can predict the next one by consulting Lorenz's map.

The second lesson is that the order in chaos involves nonlinear relationships, in which causes can produce disproportionate effects. The curve in the Lorenz map is not a straight line. It's an upside-down V. In the next lecture, we'll start to see why nonlinearity is so tremendously important (perhaps the most important idea in this course).

Unfortunately, as with Poincaré, Lorenz's discoveries about chaos fell like the proverbial tree in the forest, making a sound that no one heard. There were two reasons for the tepid reaction. First, the paper was published in a meteorology journal that few outsiders would read. Additionally, insiders didn't know what to make of it. Lorenz had butchered the hallowed equations for convection, hacking off important terms to reveal the essence of chaos but making his results seem artificial, or at best, hard to interpret.

In general, the scientific community was still not ready for chaos. Lorenz's seminal 1963 paper was cited only about once a year for its first decade. However, by around 1975, the paper started to take off, averaging about a hundred citations a year in the 1980s. In the next lecture we'll see what sparked the fire. ■

Essential Reading

Stewart, *Does God Play Dice?* chap. 7.

Supplementary Reading

Strogatz, *Nonlinear Dynamics and Chaos*, sec. 9.4.

Internet Resource

Play with Lorenz's iterated map at this site: http://www.aw-bc.com/ide/idefiles/media/JavaTools/lrnzzmax.html.

Questions to Consider

1. Try to extract an iterated map from your bank account records. Make a graph of (x, y) pairs, showing your balance at the end of one month (y) as a function of the balance at the end of the previous month (x). Include as many pairs of months as you can, to see if a trend emerges on the graph. Do the data fall on a straight line, on a curve, or do they form an amorphous blob?

2. Suppose Lorenz's system had been cycling repetitively rather than fluctuating chaotically. What would he have found when he graphed the current maximum of z as a function of its previous maximum?

How Systems Turn Chaotic

Lecture 9

In common parlance, of course, *bifurcation* means a fork or a splitting. Mathematicians have borrowed the term to refer to situations where a dynamical system branches off into a new type of behavior, a new qualitative manner of behaving in the long run as one of its parameters is varied. That doesn't happen here, but it will happen in the logistic map.

Our detective story builds to its climax in the 1970s, with an unprecedented convergence of scientific disciplines. Researchers in mathematics, ecology, and fluid mechanics (who would normally have little to say to each other) found themselves asking the same question. Whether they were studying the dynamics of iterated maps, insect populations, or water flowing through a pipe, they all wondered how their systems made the transition from order to disorder, from stable equilibrium to regular oscillations—to wild, seemingly random fluctuations. In other words, how did a system turn chaotic? This is a new question for us in this course. We're no longer asking about the properties of chaos; we're asking about the *transition* to chaos. That turns out to be a marvelously fruitful question, because we'll see that the process of going chaotic has certain universal features, independent of whatever it is that's behaving chaotically. In other words, different things go chaotic in the same way.

Let's begin with the simplest possible iterated map. It describes the growth of money compounded annually at a constant interest rate. The "map" is the mathematical function that tells us how much money we have a year from now, given how much we have today. When graphed, the relationship is a straight line, so we say this map is "linear." By "iterating" the map, we can project into the future as far as we like.

To illustrate the transition from order to chaos, as well as the ideas of bifurcation and nonlinearity, we turn now to a famous iterated map known as the *logistic map*. Suppose you're a population biologist studying the agriculturally important problem of seasonal outbreaks of a certain insect pest. The logistic map tells you how the total insect population this year can be related to the population of insects in the next generation a year f rom now.

The graph of this relationship is hump-shaped, given by the parabola $y = rx(1-x)$, where the parameter r is a constant that reflects the population's intrinsic growth rate. The idea is that if last year's generation was small, this year's will tend to be larger (because there were plenty of crops for all the bugs to eat). But if last year's generation was big, overcrowding and starvation may cause this year's generation to be small.

Iterating the map allows you to predict the generation size many years ahead. What happens in the long run depends on the growth rate, or equivalently, how steep the hump is. For low growth rate, the insects die out completely. As we gradually turn up the steepness, nothing different happens until we cross a threshold, or *bifurcation point*. Then the dynamics change qualitatively: The population approaches an equilibrium level that stays constant from year to year.

Turning up the growth rate even more yields a series of dramatic bifurcations. First the equilibrium splits into a 2-year cycle of boom and bust. Then that 2-year cycle splits into 4-year cycles; then 8, 16, 32, and so on. The cycle length becomes infinite at a critical value of growth rate. Just beyond that point, the yearly population fluctuates wildly. By increasing the growth rate, we drove the logistic map from order to chaos. There's a lot more to this story, but let's defer the details until Lecture 10.

Although the logistic map originated in ecology, it holds lessons for all of science. What the logistic map shares with so many other systems is its *nonlinear* character. Simply put, the map isn't a straight line. It's bent and hump-shaped. That's why we couldn't predict its dynamics with algebra and had to use a computer. The same difficulty afflicted Poincaré in the three-body problem; it's also governed by nonlinear equations. So are Lorenz's weather model and convection model. Nonlinearity giveth, and nonlinearity taketh away. It makes chaos possible but formulas impossible.

"Nonlinearity" sounds repulsive, I know, but you must try to fall in love with the concept. It's the most important idea in the science of chaos.

Mathematician Stanislaw Ulam quipped that calling a system nonlinear was like describing zoology as the study of "non-elephant animals." Most animals are not elephants, and most systems are not linear! What's so special about linearity? It implies the whole is merely the sum of the parts. The banking problem gives a simple numerical example of this.

Nonlinearity arises whenever the whole is more (or less) than the sum of the parts. Whenever parts of a system compete or cooperate, and don't just add up, you have nonlinearity. The nonlinearity of the logistic map reflects the insects' competition for food and mates when they're overcrowded. Everyday examples of nonlinearity are the pleasures of music, the dangers of drug interactions, and the life-saving power of combination therapy for HIV.

Mathematician Stanislaw Ulam quipped that calling a system nonlinear was like describing zoology as the study of "non-elephant animals." Most animals are not elephants, and most systems are not linear!

For hundreds of years after Newton, physicists shunned nonlinearity. They didn't need it in electrical or civil engineering, or in quantum mechanics. Their mathematical methods couldn't handle it anyway. When nonlinearity was unavoidable, they tamed it by "linearizing" it (sweeping it under the rug).

But by the 1970s, a few mavericks in every field were ready to tackle nonlinearity, and with it, chaos. Which raises the question: Why *then*? Computers were part of it. Scientists could finally see chaos right before their eyes; it wasn't abstract anymore (as it was in Poincaré's era). Also, the term itself was catchy. "Chaos" was christened by the mathematicians T. Y. Li and James Yorke in 1975 in a famous paper about iterated maps.

In 1976, the theoretical biologist Robert May published an influential review article in which he made an impassioned plea that it was time to start studying nonlinear systems. He wrote, "Not only in research, but also in the everyday world of politics and economics, we would all be better off if more people realized that simple nonlinear systems do not necessarily possess simple dynamical properties.” ■

Essential Reading

Gleick, *Chaos*, 59–80.

Stewart, *Does God Play Dice?* 154–64.

Supplementary Reading

May, “Simple Mathematical Models.”

Peak and Frame, *Chaos Under Control*, chap. 5.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 10.

Internet Resource

Explore the dynamics of the logistic map yourself. Go to http://www.student.math.uwaterloo.ca/~pmat370/JavaLinks.html to find some relevant Java applets.

One of the simplest to use is this one: http://www.geom.uiuc.edu/~math5337/ds/applets/iteration/Iteration.html.

Questions to Consider

1. Pick a number between 0 and 1000, and type it into a pocket calculator. Keep pressing the square root button repeatedly. What happens? Why?

2. What are some other familiar, real-world examples of linear and nonlinear systems?

Displaying How Systems Turn Chaotic

Lecture 10

In part, we want to study the orbit diagram because it's … breathtakingly beautiful. … [But] it's [also] a Rosetta stone for chaos in the real world. … This is a way to decode the chaos. … Mankind has been trying to do [this] for thousands of years, and we finally got it, at least for a certain kind of chaos.

To grasp the most surprising result in the entire subject of chaos (coming in Lecture 11), we first need to deepen our investigation of the logistic map. As we saw in the last lecture, the logistic map is a simple mathematical model with very complicated dynamics. Originally a model of the year-to-year fluctuations in an insect population, it's now a paradigm of chaos more generally. We found we could make this system behave in wildly different ways—from maintaining a stable equilibrium level, to oscillating in boom-and-bust cycles, and finally to bouncing around chaotically—just by tuning a single parameter.

Our goal in this lecture is to display all this fantastic richness (and more) in a single image known as the "orbit diagram." It's like a family portrait for the logistic map. But instead of a lot of cousins standing side by side, this family portrait shows all the different things the logistic map can do as we vary its growth rate. As mentioned, we want to study the orbit diagram not only because it's so beautiful, but also because it's a Rosetta stone for chaos in the real world, as we'll see in the next two lectures. By deciphering its secrets, we'll be able to make a series of powerful predictions about the route to chaos in everything from electronic circuits to dripping faucets.

So let's see how the orbit diagram is constructed. You can continue to think of it as a family portrait of mathematical cousins standing side by side, or it may be better to think of it as a series of vertical slices through some complicated object (like a CAT scan). By stacking the slices together, we reconstruct the whole picture. To create each slice, choose the associated value of the growth rate and then iterate the logistic map thousands of times with a computer. We're trying to see how the population is eventually going to behave. Since we only care about the long run, let's disregard what happens early on, while the system is still settling down to its ultimate behavior. After waiting long enough, we plot all the different values of the population that occur. This amounts to plotting the attractor for this slice of the diagram.

It turns out that the diagram divides naturally into two parts: an orderly part and a chaotic part. Let's begin with the simpler, orderly part. This part of the diagram looks like a tree. A trunk rises up from the ground and splits into two branches, each of which splits into two more, and so on, ad infinitum. The trunk signifies a population at equilibrium. As we increase the steepness parameter, the trunk splits (bifurcates) into a 2-year cycle of high and low populations—recall, we saw this kind of cycle in Lecture 9. Subsequent splittings represent cycles of length 4, 8, 16, etc. The period of the cycle doubles at each bifurcation.

There are two striking features of the tree. First, it looks self-similar; each portion of the tree resembles the whole. This self-similarity is one of the defining features of a *fractal* (to be discussed in Lectures 13 and 14). Additionally, the twigs of the tree shrink with each branching. As the structure becomes infinitely bifurcated, it seems to accumulate at an impenetrable wall, sometimes called the *accumulation point*. When theoretical biologist Robert May first encountered this amazing image, he was so bewildered that he left a note on a corridor blackboard for his graduate students, asking them, "What the Christ happens beyond the accumulation point?"

The whole diagram repeats in miniature, infinitely often. And … each of those mini-diagrams contains its *own* windows, with their own *mini*-mini-diagrams, on and on, like a hall of mirrors. We see all this from the simplest nonlinear system.

The answer is: chaos. But as you should expect by now, it's not formless—there's incredible order intermingled with it. Smooth, dark tracks run through the chaos. Vertical windows interrupt the chaotic scatter. Each harbors a cycle of some length. For example, there is a big window containing period-3 cycles and a smaller one containing period-5 cycles. Most astonishing of all, each window ends with its own little copies of the entire orbit diagram. The whole diagram repeats in miniature, infinitely often. And of course, each of those mini-diagrams contains its *own* windows, with their own *mini*-mini-diagrams, on and on, like a hall of mirrors. We see all this from the simplest nonlinear system. Just think what nature is capable of!

In the next lecture, we'll see there are further patterns in this picture—very precise numerical patterns. What's important about them is their universality.

The same patterns were found to occur in other maps and in differential equations, suggesting that they might also occur in real-world phenomena. In this way, the logistic map becomes more than a computer game, and the orbit diagram, more than a pretty picture. Together they provide a raft of scientific predictions, with great power for making sense of certain forms of chaos in the natural world. ■

Essential Reading

Gleick, *Chaos*, 59–80.

Stewart, *Does God Play Dice?* 154–64.

Supplementary Reading

Peak and Frame, *Chaos Under Control*, 169–76.

Strogatz, *Nonlinear Dynamics and Chaos*, sec. 10.2.

Internet Resource

Explore the orbit diagram yourself. A very nice Java applet is here: http://math.bu.edu/DYSYS/applets/bif-dgm/Logistic.html. Try to find the mini-orbit diagrams at the end of a big periodic window.

Questions to Consider

1. In what ways is the orbit diagram simple? In what ways is it complex?

2. Discuss more generally what it means for a mathematical or scientific object to be simple or complex.

Universal Features of the Route to Chaos

Lecture 11

[4.6692… and 2.5029]… are to chaos what pi (π) is to circles and periodicity. They are new fundamental constants of mathematics. They are the fundamental constants of chaos. … They're [also] … fundamental constants of nature because chaos is in nature, and this description we're giving, as we will see in the next lecture, applies to nature.

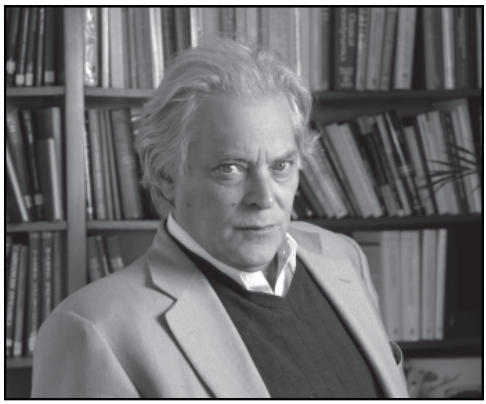
We're now ready to behold the most incredible and disconcerting result in the whole subject of chaos. We'll see that what happens in the logistic map (from Lectures 9 and 10) happens *universally* in a very wide class of systems. In fact, the logistic map displays such universal features en route to chaos that we have good reason to expect they must also occur in nature. Our goal is to understand exactly what we mean by *universality*, why it matters, and why its discovery was the most stunning breakthrough in the new science of chaos.

Let's begin by explaining the feature known as the U-sequence ("U" for "universal"). Take a look at the orbit diagram (from Lecture 10). Focus on the prominent vertical stripes. These are the "periodic windows" for the logistic map. They represent parameter values where the system behaves periodically, not chaotically. In other words, the map generates numbers that repeat after every 3, 4, 5, or some other whole number of iterations. This kind of attractor is called a *stable cycle*. The biggest window is the one for period-3 cycles (meaning those cycles that repeat every 3 steps). To the left of it is the window for period-5 cycles, and even farther left, period-6.

The sequence seems strange: 6, 5, 3. What's the pattern? Let's step back a bit and look at *all* the stable cycles, including those in the tree portion of the diagram. Since there are infinitely many stable cycles, let's only look at the ones with short periods, say period 6 or smaller. Then, starting from the left of the diagram and moving to the right, the periods of the stable cycles appear in the following sequence: 1, 2, 4, 6, 5, 3, 6, 5, 6, 4, 6, 5, 6. (Most of the windows after period 3 are invisible at this scale, but they are there.)

Now for the shocker: This same weird sequence of numbers occurs for *any* iterated map, as long as it is hump-shaped like the logistic map. What's so amazing about this is that only the rough shape of the map matters. Its precise algebraic formula is totally irrelevant—even though changing the formula would certainly alter all the numbers that come out of it. Somehow, the periodic windows don't care about those details; geometry trumps algebra here.

Now we'll look at an even more incredible kind of universality, one that *does* involve specific numbers. (That's a selling point, because scientists regard numbers as strong predictors.) Focus on the tree part of the orbit diagram. Look at successive wishbones in the picture and measure their heights and widths. The wishbones shrink in both directions as we approach the onset of chaos. What's the rule governing how *quickly* they shrink, in both directions? Each wishbone is about 4.7 times smaller (measured sideways) and 2.5 times smaller (measured vertically) than the one to the left of it. Roughly the same shrinkage factor applies all along the tree. As you approach the onset of chaos, the shrinkage factors converge to very specific, very peculiar numbers: 4.6692… and 2.5029… .



American theoretical physicist, Mitchell Feigenbaum (b. 1944) used a pocket calculator to uncover a stunning kind of universality in chaos--that diverse systems go chaotic in the same way.

Here comes the universality. Suppose you redo the measurements for another hump-shaped map whose algebraic form is completely different from the logistic map's (for instance, the hump of a sine wave instead of a parabola). Then the numerical details of the tree will change. The widths of the wishbones will change. But the shrinkage factors again turn out to be 4.7 horizontally and 2.5 vertically. And near the onset of chaos, they converge to 4.6692… and 2.5029…, exactly the numbers seen before!

This quantitative form of universality was discovered by physicist Mitchell Feigenbaum (b. 1944), a colorful character. Currently a Professor of Mathematics and Physics at Rockefeller University, he has intense eyes, a mischievous grin, and the swept-back hair of a Romantic composer. His seminal discoveries came in 1975 while he was working at Los Alamos in the Theoretical Division. His working habits were odd, even for T-Division: He roamed the streets in the middle of the night; lived on cigarettes and soda; and kept to a 26-hour day, his sleep-wake schedule rolling in and out of phase with the rest of society every few weeks. His discoveries about universality were published in 1978, but only after several journals rejected them.

The implications of Feigenbaum's ideas are shocking. … The logistic map has no scientific content, no laws of nature built into it. It seems like pure numerology.

Feigenbaum predicted that the universal features he'd discovered in iterated maps must also occur in nature. After all, he'd found the same patterns in differential equations. And since the laws of nature are expressed by differential equations, the universal numbers 4.669… and 2.502… should appear in real-world chaos too. The only caveat was that the system had to arrive at chaos via a cascade of period-doubling bifurcations, just as in the logistic map. Otherwise all bets were off.

The implications of Feigenbaum's ideas are shocking. What makes them so unbelievable is that the logistic map has no scientific content, no laws of nature built into it. It seems like pure numerology. This breaks the rules of how to do science. Even quantum mechanics and relativity had not dispensed with the idea of basing predictions on fundamental laws of nature. And yet, miraculously, subsequent experiments on a variety of real systems conformed to Feigenbaum's predictions (which we will see in Lecture 12). It was as if the old Pythagorean dream had come true. The universe is not made of earth, air, fire, or water—it is made of number.

Another startling quality of Feigenbaum's work was its cross-disciplinarity. He'd applied a Nobel Prize-winning method in condensed-matter physics to a math problem a child could play with on a calculator—and from this came predictions about the route to chaos in heart cells, electronic circuits, and convecting fluids! ■

Essential Reading

Gleick, *Chaos*, 157–87.

Stewart, *Does God Play Dice?* chap. 10.

Supplementary Reading

Cvitanovic, *Universality in Chaos*.

Strogatz, *Nonlinear Dynamics and Chaos*, sec. 10.6.

Questions to Consider

1. Does the universality of Feigenbaum's predictions make them powerful (because they apply so widely) or weak (because they are so generic and unspecific)?

2. After Feigenbaum achieved his breakthroughs in the late 1970s, he was rarely heard from again and has published relatively little. Why might this be? Enlarge the discussion to include other creative individuals who did the same thing, from scientists to writers, artists, musicians, filmmakers, and inventors.

Experimental Tests of the New Theory

Lecture 12

When [a team lead by Harry Swinney at The University of Texas] pushed the system past the accumulation point … to the onset of chaos, they could see chaos … but more than that, the chaos was interrupted by periodic windows. It's incredible. … They saw that in reality. … And the order of the periodic windows was the magic universal sequence, 6 came before 5 came before 3. All in all, a great confirmation of the theory.

The moment of truth for the science of chaos comes in the early 1980s, with the first experimental tests of the new theory. In case after case, the theory's predictions were confirmed. It was a heady time. Researchers were starting to make sense of the borderland between order and disorder. But as with any theory, the new ideas apply only in certain restricted settings—here, for systems with only a few variables, not millions. The lecture concludes with a balanced assessment of where the theory works and where it breaks down.

One elegant experiment involved an oscillating chemical reaction. Such reactions are a favorite demonstration in chemistry classes, because they change color with clocklike regularity. Under some conditions, the reaction can switch from clockwork to chaos.

Harry Swinney (b. 1939), a physicist at The University of Texas, led a team that tracked the chaos by recording the reaction's bromide ion potential as a function of time. The peaks and valleys in the time series vary erratically. But is this genuine deterministic chaos or merely randomness caused by sloppy experimental technique?

To find out, they graphed their data in state space, and out popped a strange attractor. Remarkably, it looks like one wing of Lorenz's butterfly attractor (discussed in Lecture 7).

Within the chemical strange attractor, the dynamics were animated by an iterated map, just as Lorenz's had been. The map has a single hump, like

Lorenz's map and the logistic map (from Lectures 8 and 9). When the experimenters varied the flow rate of chemicals feeding through the reaction, they observed the period-doubling route to chaos, just as in the logistic map. Within the chaotic region, the system also has periodic windows, and they occur in the order predicted by the universal sequence (from Lecture 11). All in all, a great confirmation of the theory!

Other researchers did similarly careful experiments on convecting fluids and electronic circuits. They confirmed the universal number 4.669… that Feigenbaum had predicted for the period-doubling route to chaos.

But there are important caveats. First, the experiments are difficult. You can rarely measure more than four or five period doublings, because noise in the experiments obliterates the fine structure of the high-period cycles, which makes it hard to tell precisely when a bifurcation has occurred. Also, it's hard to vary the control parameter precisely enough. (Recall that the spacing between bifurcations shrinks roughly 5-fold each time, so each new data point requires a 5-fold improvement in the experimenter's ability to tune the parameter.)

Most importantly, tremendous care must be taken to suppress all other possible instabilities that could occur. Otherwise, when you drive up the flow rate or turn up the temperature, the system may go unstable in ways you don't want. It often starts to vary in *space* as well as time. For example, in the chemical experiment, the system is continuously stirred so that it can't set up any spatial patterns. In the convection experiments, scientists used liquid mercury instead of water; the advantage was they could keep the convection rolls perfectly straight by aligning them to a constant magnetic field. Otherwise the rolls would start to wiggle and eventually degenerate into seething turbulence. Feigenbaum's theory can't handle such complications. It only applies to systems following a nice, stately period-doubling route to chaos.

Still, back in 1983, all this was a remarkable triumph for the emerging science of chaos. None of these phenomena could have been understood just a decade earlier. Moreover, no one would have even thought to ask the questions. And that is perhaps the greatest triumph of all.

To assess the strengths and limitations of chaos theory more clearly, consider where it resides on a metaphorical "thermometer of complexity." Just as on a real thermometer, the coldest temperatures are at the bottom. This is where order lives, the realm of regular, well-behaved systems that Newtonian science mastered and that many great scientists are continuing to study today. Turn up the heat a bit by stressing the system or forcing it to feed back on itself. Where order first begins to break down, we enter the realm of chaos—but not total chaos. Instead, we see a chaos that lives much closer to regularity than mayhem.

Next we come to systems that vary wildly in space as well as time. Think of turbulence in a roiling pot of water, or fibrillation in a quivering heart. Still, we have hope of understanding them—the laws of fluid mechanics are known, and the laws of cardiac electrophysiology are becoming clearer every day.

Finally, we reach the boiling point—the most complex, unpredictable systems we can imagine, with billions of interconnected variables. Think of the global economy, human society, the brain. Here we don't know the laws. Plus they might be changing or nonexistent.

The point is that chaos theory lives in a very cool part of the thermometer. In their enthusiasm, some people in the '80s tried to apply it to much hotter systems—the stock market, or turbulence, or warfare between nations. Such overreaching provoked a backlash that also went too far. Finally, notice that as the mercury rises, we also retrace the historical development of chaos theory and point toward the problems at the cutting edge.

In their enthusiasm, some people in the '80s tried to apply [chaos theory] to much hotter systems—the stock market, or turbulence, or warfare between nations. Such overreaching provoked a backlash that also went too far.

In the next half of the course, we'll look at the most exciting advances in chaos theory that have occurred since the 1980s, including some recent ones that take us much higher on the thermometer of complexity. But first, to prepare ourselves for the wonders that lie ahead, we need to become better acquainted with fractals, the infinitely intricate shapes that we've already encountered on several occasions. ■

Essential Reading

Gleick, *Chaos*, 191–211.

Stewart, *Does God Play Dice?* chaps. 9–10.

Supplementary Reading

Cvitanovic, *Universality in Chaos*.

Roux, Simoyi, and Swinney, "Observation of a Strange Attractor."

Strogatz, *Nonlinear Dynamics and Chaos*, sec. 10.6.

Questions to Consider

1. What observations could you make to distinguish between a deterministic but chaotic system and a system that is random?

2. Do you believe that massively complex systems like the brain, the global economy, and human society obey natural laws in the same sense that pendulums and planets do?

Fractals—The Geometry of Chaos

Lecture 13

This baker may … [put] a little pat of butter in [the pastry]. That pat … represents nearby initial conditions. That is, there is a tiny blob consisting of states that are all close together, particles close together. … What would happen if I were to do my process of rolling, folding, rolling? … It will cause mixing. But mixing is chaos.

We've now reached the midpoint of the course, and it's time to change gears slightly. The next six lectures are devoted to fractals—beautiful, intricate shapes with endless self-similar structure. We'll see that fractals *are* beautiful, yes, but they're also much more than that: They're crucial to the science of chaos. One could even say that they're the geometry of chaos. And they're changing the way we think about complexity in all its forms.

By that, I mean that fractals are useful all the way up and down the metaphorical "thermometer of complexity" (from Lecture 12). For example, we'll see how fractals shed light on such poorly understood phenomena as earthquakes and stock market fluctuations. Both are illuminated by fractal statistics—a very different kind of statistics from the orthodox kind you might have learned about elsewhere. Thus, fractals are going to play an essential part in the science of the 21st century as we strive to unravel systems of ever-increasing complexity.

Although fractal geometry might seem like a new topic for us, it really isn't. We already encountered fractals in Lecture 10, when we noticed the self-similarity of the fig tree, with its infinitely bifurcated structure. And later in that same lecture, we spotted the incredible miniature copies of the entire orbit diagram at the end of each periodic window. Then, in Lecture 7, we puzzled over the infinitely many layers of Lorenz's strange attractor, all packed together like sheets of mica. But in all those cases we mentioned fractals en passant, whereas in the next few lectures we're going to put them at the center of our attention.

First, we consider the cause of the infinitely layered structure of strange attractors, which is the same as the cause of chaos: a repeated process of stretching and folding in state space. Consider an analogy with the way that pastry is made, the very flaky kind with thousands of paper-thin layers, as in croissant or strudel. Think of state space as being like dough. The particles of flour represent all the possible states, all possible initial conditions.

The differential equation (or iterated map) does roughly the same thing to state space that the baker does to the dough. Rolling the dough flattens it. In the analogy, this compresses state space into a sheet that later forms the paper-thin layers of the attractor. Rolling also stretches the dough lengthwise. This has the effect of spreading nearby bits of dough apart—which is analogous to chaos (two nearby initial states diverge rapidly toward different fates). Finally, repeated folding brings distant parts of the sheet close together, creating many flaky layers.

This intuition about stretching and folding occurred independently to three scientists in different fields (math, chemistry, and astronomy), all of whom played important roles in chaos theory.

This intuition about stretching and folding occurred independently to three scientists in different fields (math, chemistry, and astronomy), all of whom played important roles in chaos theory. The first person to have the idea was Stephen Smale (b. 1930), one of the world's greatest living mathematicians. If we were doing a pure math course on chaos theory, he would be one of the heroes. In the 1960s, he invented something called "the horseshoe mapping" (which stretches and folds a rectangle into a horseshoe shape, just like the baker does to the dough), and he used it to prove very important theorems about chaos.

Next came Otto Rössler (b. 1940), a chemist. Working in the mid-1970s, he knew about Lorenz's work and wanted to concoct an even simpler example of a chaotic differential equation. He constructed an example that has just one nonlinear term to Lorenz's two. Rössler's inspiration came from watching a taffy-pulling machine as it was stretching and folding taffy. The strange attractor produced by Rössler's system has been found in many real systems. (In fact, it was the strange attractor for the chaotic chemical system we discussed in Lecture 12—it looks like one wing of Lorenz's butterfly.)

Finally, astronomer Michel Hénon (b. 1931) demonstrated that stretching and folding generate a hierarchical structure that is "self-similar." Recall, self-similarity means that small pieces of a structure resemble the whole. It's one of the defining features of a fractal. Hénon devised his mathematical version of the pastry-making process because he wanted to visualize the layered structure of Lorenz's strange attractor. Hénon's mapping takes a solid rectangle and then flattens it, stretches it, and folds it into a horseshoe shape. If we iterate long enough, we should see the ghost coming out of the mist. The object that appears is the Hénon attractor. From a distance it resembles a boomerang. Zooming in reveals three sets of parallel lines, arranged in groups of 1, 2, and 3 thinner lines per set. Further magnifications show that this structure repeats endlessly, at smaller and smaller scales. It is the basic structure of strange attractors—what Lorenz meant by an "infinite complex of surfaces."

Self-similarity is actually a very old and universal idea (now being mathematized by fractals). Artists with a keen eye for nature have long been fascinated by it. Think of Leonardo da Vinci's sketches of turbulent whirlpools, or Hokusai's woodblock print of *The Great Wave off Kanagawa*, or Escher's visual explorations of infinity in a confined space. Once you start to look for fractals, you'll notice them everywhere in the world around you, and even inside you, from mountains and clouds to the branching patterns of your own blood vessels.

In the next five lectures, we'll define fractals more precisely. We'll also see how they illuminate a diverse set of complex phenomena in nature, technology, business, and art. ■

Essential Reading

Gleick, *Chaos*, 83–118 and 144–53.

Mandelbrot, *The Fractal Geometry of Nature*.

Schroeder, *Fractals, Chaos, Power Laws*, chaps. 3, 11.

Stewart, *Does God Play Dice?* chap. 11.

Supplementary Reading

Liebovitch, *Fractals and Chaos Simplified for the Life Sciences*, pt. I.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 11.

Internet Resource

Play with Hénon's mapping and strange attractor here: http://www.cmp.caltech.edu/~mcc/Chaos_Course/Lesson5/Demo1.html.

Questions to Consider

1. What other artists have touched on fractals in their work?

2. Does mixing always rely on chaos? Discuss the process of shuffling cards or stirring milk into coffee in these terms.

The Properties of Fractals

Lecture 14

The key point is that fractals behave oddly when you magnify them.

Last time we discussed fractals as the embodiment of chaos. Now we look at fractals more generally, beginning with their two most distinctive features (their inexhaustibility of structure and their self-similarity) and then touching on some larger issues:

- Where else do fractals arise, outside of chaos theory?
- How did the science of fractals come into being?
- Answering these questions will help us appreciate the unique relationship between fractals and chaos, as well as the place of fractals in the broader scientific landscape.

To contrast fractals with familiar (non-fractal) shapes, imagine looking at a fractal under a microscope. As you turn up the power, inexhaustible fine structure is revealed. In contrast, Euclidean shapes (like circles or spheres) start to look featureless as you zoom in. Furthermore, the small features of a fractal resemble the larger ones. In this sense the structure is "self-similar." For an edible example, consider a head of Romanesco broccoli, whose florets look like mini-Romanescos.

Strictly speaking, however, only mathematical fractals are truly inexhaustible and self-similar. Real objects (like the Romanesco) are only approximately fractal. The repetition of structure isn't perfect, and it stops below some smallest scale. Thus, the issue is the range of scales over which an object can be usefully regarded as fractal. The wider the range and the more faithful the self-similarity, the more fractal it is.

Self-similarity can be viewed as a kind of symmetry. An object has a symmetry if its stays the same despite a change. For example, a sphere has rotational symmetry—it stays the same if you rotate it. Our bodies have mirror symmetry—the left side looks like the right. Likewise, a fractal has scale symmetry—it stays the same if you magnify it. This type of symmetry hasn't been studied much until recently.

The concept of *scales* is important for distinguishing fractals from non-fractals. Ordinary objects tend to have just a single characteristic scale, given by the size of their smallest features. Below that scale they look bland and unstructured. In contrast, fractals vary over such a wide range of scales that they are described as being *scale-free*. For example, geologists are taught to place a hammer or a coin next to any craggy rock formation before photographing it. Otherwise, without these clues to set the scale, the viewer will be unsure whether the image shows a mountain, a cracked pebble, or a magnified speck. Likewise, the jagged charts of stock prices look remarkably similar, whether shown minute-by-minute over the span of a day, or day-by-day over the span of a year. There's something liberating about being scale-free. Maybe that's why we find such beauty in the appearance of clouds, cracked rocks, trees, and frost on a window pane.

But the scale-free structure of fractals also poses confusing challenges of measurement. Suppose you were asked to the measure the surface area of a football field. It's 100 yards long and 50 yards wide, so you answer, "5000 square yards." But that neglects the surface area of each blade of grass. In this problem, there are (at least) two scales—that of a human and that of a blade of grass (which, if an ant were making the measurement, would be the more obvious scale). The point is that in problems with multiple scales, it's meaningless to report a single answer. The answer you get depends on the resolution you use. The right approach is to quantify exactly *how* the answer depends on the resolution. We'll elaborate on this in the next lecture, using Norway's coastline for illustration.

As the examples of broccoli, coastlines, stock prices, and mountains should suggest, fractals have real-world applications far beyond the strange attractors and other abstractions we encountered when visualizing chaotic dynamics. The ideas of fractals have now been incorporated into virtually every branch of science. But the subject was largely unknown before the mid-1970s, when it was synthesized, popularized, and christened by Benoit Mandelbrot (b. 1924), a polymath then working at IBM and now associated with Yale. Mandelbrot's interests ranged from commodity price fluctuations to computer graphics, and from the structure of galaxies to the firing patterns of nerve cells. Besides his own pioneering contributions, he drew on older, scattered work from an amazing range of disciplines—everything from pure mathematics to architecture—and thereby created the new field of "fractal geometry," which he named for all that is fragmented, fractured, broken, and irregular. There's an irony here. Earlier mathematicians had created fractals as pathological counterexamples, monsters to chasten their intuition of what a curve or surface could be like. These examples were regarded as weird and unnatural. Yet nature has been using fractals all along!

Is fractal geometry part of chaos theory, or vice versa? They are parallel developments. Both deal with irregularity. But fractals are mainly about irregularity in *space*; chaos is about irregularity in *time*. Each developed independently, but also with important intersections (such as the fractal shapes that arise in the visualization of chaotic systems, as we've discussed).

Fractals are mainly about irregularity in *space*; chaos is about irregularity in *time*.

Still, that's not why the public associates chaos with fractals. Their association in popular consciousness is something of a historical accident. Both subjects became media darlings at the same time. Chaos theory was confirmed experimentally in the early 1980s, just before the first spectacular images of fractals began to appear. And perhaps their simultaneous appearance was a reflection of a zeitgeist in the world of science, a readiness to finally confront the disorderly, jagged side of nature and to search for the hidden patterns within it. ■

Essential Reading

Gleick, *Chaos*, 83–118.

Mandelbrot, *The Fractal Geometry of Nature*.

Schroeder, *Fractals, Chaos, Power Laws*, chaps. 1, 10.

Stewart, *Does God Play Dice?* chap. 11.

Supplementary Reading

Liebovitch, *Fractals and Chaos Simplified for the Life Sciences*, part I.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 11.

Questions to Consider

1. Some objects found in nature are Euclidean (like crystals) and others are fractal (like ferns). Give several examples of each type.

2. Real objects are fractal only over a limited range of scales. What other scientific concepts hold only over a limited range? Is this true of all scientific concepts?

A New Concept of Dimension

Lecture 15

First of all, scientists use fractal dimensions all the time to characterize the roughness of highly irregular objects of all sorts: coastlines, tumors, galaxies, even the folded surface of the human brain.

Perhaps the strangest thing about a fractal is its dimensionality. Common sense tells us that a line is 1-dimensional, a surface is 2-dimensional, and a solid is 3-dimensional. But for the highly corrugated shapes typical of fractals, our usual ideas about dimensionality break down. In this lecture we'll explain how to generalize the concept of dimension so that it can handle fractals as well as ordinary, Euclidean shapes. The weird but inescapable conclusion is that fractals are so convoluted that they're not 1-, 2-, or 3-dimensional but somewhere in between, such as 1.26-dimensional! This might seem like an esoteric subject, but it's worth studying for several reasons. Scientists use fractal dimensions to characterize the roughness of highly irregular objects. Before this concept was developed, there was no way to quantify different degrees of jaggedness. We need to understand how fractal dimensions are calculated because they'll come up again later, for instance when we discuss the controversy about the authentication of some recently discovered paintings alleged to be by Jackson Pollock (Lecture 18). And it's fascinating to think about something as basic as dimension in a new way.

The easiest place to start is with objects that are perfectly self-similar. Such objects are made up of copies of themselves, down to arbitrarily small scales. For example, consider a square patch of carpet. It can be divided into many smaller squares, each a tiny copy of the original. For instance, we can chop the carpet into 4 smaller squares by halving each of its sides. Or 9 smaller squares, by dividing each side by 3. Now we're getting close to seeing how dimension enters into it. Notice that $4 = 2 \times 2$, and $9 = 3 \times 3$. Writing this as a number raised to a power, we'd say $4 = 2^2$, and $9 = 3^2$. That power of 2 is no accident. It occurs because the square patch is 2-dimensional. Likewise, slicing a solid cube in half on each side yields 8 smaller cubes. Here, $8 = 2 \times 2 \times 2 = 2^3$, and the power of 3 occurs because the solid cube is 3-dimensional. The power (or exponent) in such an equation tells us the object's dimensionality. If we use the symbol m for the number of smaller copies, r for the reduction factor in each direction, and d for the dimension, we can summarize all this with the equation $m = r^d$.

Now play the same game with a self-similar fractal. We'll find that the exponent—which we still interpret as the object's dimension—is no longer a whole number. Consider the Koch curve, an exquisitely thin and crinkly filament that looks somewhat like the perimeter of a snowflake. It's constructed by iterating a simple geometrical growth rule. Start with a straight line segment. Then, whenever you see a line segment, take out its middle third and replace it with a V shape, provided by the top 2 legs of an equilateral triangle. Keep doing that ad infinitum. The limiting shape is the Koch curve.

The Koch curve refers to the edge of the snowflake, not the area inside. Here's the amazing thing: You can split the Koch curve into 4 perfect copies of itself, each shrunken 3-fold, as if reduced by a photocopier. Incredibly, when the reduction factor is $r = 3$, we get $m = 4$ copies. So, from the equation $m = r^d$ we mentioned earlier, the dimension of the Koch curve satisfies $4 = 3^d$. We can solve for d by using logarithms (don't worry if you've forgotten them). The result is $d = \log 4 / \log 3 = 1.26\ldots$.

Thus the Koch curve is more than 1-dimensional, but less than 2! Like a line, it is infinitely thin and covers no area. But unlike a line, the path between any 2 of its points is infinitely long. You could never walk along it. It's so crinkly that you couldn't even get started. So the Koch curve lives in a bizarre netherworld—more than a line, less than an area. Its dimension of 1.26 reflects that in-between status.

Scientists have generalized the concept of dimension even further, to objects that aren't exactly self-similar (as most real objects are not). For example, suppose we want to measure how long Norway's coastline is. This is difficult, because there are many fjords, each harboring smaller inlets, and so on. The length we measure will depend on how detailed our map is. Thus, as discussed in Lecture 14, there's no one right answer; the answer depends on the resolution we use. The key is to study *how* the apparent length changes with the resolution. The variations often follow a systematic law over a wide range of scales. And that law encodes information about the coastline's fractal dimension. So the right concept here is not the coastline's length but rather its fractal dimension.

To compute the dimension, we overlay a grid on the map and count how many boxes include part of the coastline. Then we change the resolution. As the grid gets finer, more boxes are penetrated. If we graph the number of penetrated boxes against the box size and use a log-log plot, the data fall on a straight line. Its slope tells us the fractal dimension. By measuring Norway's coastline with grids of different sizes, ranging from 0.6 kilometers to 80 kilometers, Norwegian physicist Jens Feder determined that its fractal dimension is about 1.52. Norway's coast is even more rugged than the Koch curve! For comparison, the craggy west coast of England has $d \approx 1.25$, whereas South Africa's exceptionally smooth coast has $d \approx 1$.

A 1996 study of 30 normal human subjects found that the white matter surface of the brain has a dimension of about 2.3, due to its intricate folding. But don't be too impressed with yourself—the number for cauliflower is 2.8!

Fractal dimensions have now been measured for everything imaginable. For example, consider the convoluted surface of the human brain. If it were a perfectly smooth sphere, its surface would be 2-dimensional. A 1996 study of 30 normal human subjects found that the white matter surface of the brain has a dimension of about 2.3, due to its intricate folding. But don't be too impressed with yourself—the number for cauliflower is 2.8! ■

Essential Reading

Gleick, *Chaos*, 83–118.

Mandelbrot, *The Fractal Geometry of Nature*, 25–57.

Schroeder, *Fractals, Chaos, Power Laws*, 1–20.

Supplementary Reading

Feder, *Fractals*, chap. 2.

Liebovitch, *Fractals and Chaos Simplified for the Life Sciences*, 45–72.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 11.

Questions to Consider

1. What physical effects cause Norway's coastline to be fractal rather than smooth?

2. Consider the Cantor set, a famous fractal constructed as follows. Start with a line segment. Slice it into 3 equal pieces. Throw away the middle third, except for its endpoints. Then repeat this process infinitely often: Every time you see a line segment, remove its middle third. Calculate the fractal dimension of the limiting set.

Fractals Around Us

Lecture 16

The new idea in these lectures, especially in this one, is that fractals are not just static geometric shapes. They can also be erratic processes in *time*, such as price fluctuations in the stock market, bursts of data traffic on the Internet, or the unexpected rumbling of an earthquake.

Having established what fractals are and how they can be created by chaos, we devote the next three lectures to a survey of their importance in science, commerce, and the arts. The goal of this lecture is to open your eyes to fractal processes in nature and in patterns of human activity and to explain why their presence changes everything we thought we knew about risk and volatility.

How can we think of a temporal process as akin to a fractal shape? The connection is that stock prices and other erratic processes vary over a wide range of scales, both in time and in the size of their fluctuations, and in that sense they can be regarded as "scale-free" (see Lecture 14). To reinforce what we mean by scale-free structure, let's look a bit more deeply at how fractal shapes differ from non-fractal ones. To do so, we're going to draw on concepts from probability and statistics for the first time in this course. Non-fractals display features at one predominant scale. Variations may occur, but they tend to follow a bell curve distributed about some average size. Typically, the sizes of the features are "normally distributed" about a mean value.

In contrast, fractal shapes display features over a much broader range, with the small features resembling the larger ones in some way. They follow a power-law distribution, with an exponent corresponding to the shape's fractal dimension (see Lecture 15). Power laws are the algebraic expression of the scale-free structure of fractals. We focus on power laws for the rest of the lecture because they are the link between shapes and processes. They occur for all sorts of things besides geometric shapes. The idea is that events (earthquakes, stock price changes, and so on) can vary dramatically in size. Typically, large events are much rarer than small ones. In a surprising number of different settings, the relation between the size of events and their frequency follows a power law, just as the sizes of features do in fractal shapes.

Power laws have very counterintuitive properties, totally different from those taught in traditional statistics courses. That's because a power-law distribution has what's called a *heavy tail* (also known as a *fat tail* or a *long tail*). Extremely large "outliers," though still rare, are much more common for power-law distributions than for bell curves. They can have a stunning impact on a system's average behavior.

For example, suppose the world's richest man, Bill Gates, walks into a room of 100 people. His presence changes the average income in the room dramatically. But if the world's *tallest* man walks in, he won't change the average height by much. The difference is that heights are normally distributed, so the tail of the distribution drops off extremely fast (exponentially fast, which is the fastest you can get, in practice). Whereas incomes follow a power law in the long tail of the distribution (as the Italian economist Vilfredo Pareto reported in 1896, in the first scientific use of a power-law distribution). Why the difference? Heights are constrained by biology, resulting in a characteristic scale of around 5 or 6 feet, whereas incomes are unconstrained (and hence scale-free) in capitalist societies.

Likewise, data traffic on the Internet obeys fractal statistics.

Or consider some examples from finance. On October 19, 1987 (now known as "Black Monday"), the Dow Jones Industrial Average dropped by 22% in a single day. Compared to the usual level of volatility, this was a drop of more than 20 standard deviations. Such an event is essentially impossible, according to traditional bell-curve statistics. (Its probability is less than 1 in 10^{50}.) The explanation is that changes in stock prices and other financial markets (such as exchange rates) don't follow normal distributions. They are better described by fractal statistics, as Mandelbrot and his disciples have emphasized.

Likewise, data traffic on the Internet obeys fractal statistics. File sizes vary widely, from tiny e-mails to bigger photographs to enormous movies, leading to violent bursts in overall traffic. This came as a surprise in the early days of the Internet. Engineers were used to handling voice traffic over the phone network, where the statistics obey tame bell curves. The "burstiness" of Internet traffic makes it much harder to ensure reliability of the system. (Compare how often you're frustrated by delays on the Web versus how often you fail to get a dial tone.) A related complication is that the traffic fluctuates on a wide range of time scales, from hours to fractions of a second. Bell-curve models developed for voice communications don't capture this, but fractal models do a better job.

Earthquakes, wildfires, floods, and other natural hazards are also governed by fractal statistics. These processes gyrate much more wildly and frequently than one would expect on conventional statistical grounds, complicating the task of risk management in the financial and insurance industries. For example, earthquakes obey a power law, called the Gutenberg-Richter law, which relates their frequency to their size (measured by the amount of destructive energy released). The data of payouts from the insurance industry are correspondingly "bursty." Although we may never be able to predict where or when such catastrophes will occur, the hope is that a better understanding of fractal processes will provide a more rational basis for assessing their overall risk and for guarding against them. ■

Essential Reading

Schroeder, *Fractals, Chaos, Power Laws*, chap. 4.

Suggested Reading

Ball, *The Self-Made Tapestry*, 210–16.

Buchanan, *Ubiquity: Why Catastrophes Happen*.

Liebovitch, *Fractals and Chaos Simplified for the Life Sciences*, 73–105.

Mandelbrot and Hudson, *The (Mis)behavior of Markets*.

Taleb, *The Black Swan*.

Turcotte, *Fractals and Chaos in Geology and Geophysics*.

Questions to Consider

1. Give some examples of what Nicholas Taleb has christened "Black Swan" events—extremely unlikely events in history, economics, or politics that occurred nonetheless, and that changed the world.

2. In what ways would life be different if stock prices, personal incomes, earthquakes, and everything else obeyed normal (bell-curve) statistics rather than fractal (power-law) statistics?

Fractals Inside Us

Lecture 17

Whenever your body needs to send something from place to place—signals propagating along a nerve, hormones, blood, oxygen, anything that needs to be carried throughout your body to service yourself—those sorts of things are transported on a superhighway system based on fractal branching networks.

Why might evolution have favored the architecture of the fractal? If you want a plumbing system that can reach every cell in a 3-dimensional body, this is a great way to do it. Fractals also create an enormous amount of surface area in a confined space, crucial for such processes as gas exchange in the lung or absorption of food in the intestine. Finally, the blueprint for a fractal tree is simple and hence easy to encode genetically: Grow a tube, split it into several smaller branches, repeat.

The fractals inside us might also hold the key to some of life's greatest mysteries. Why do we live for about 80 years, rather than 80 seconds or 80 centuries? And why does a mouse live only a year or two, even though it's made of the same molecules and genes as we are? Despite all that biologists have learned about the mechanisms of aging, no one knows how to calculate an animal's typical life span from first principles. The rest of this lecture discusses a theory that offers insight into questions like these. It's based on the fractal architecture of all living things. The theory is provocative because it challenges us to look at life with a physicist's eyes, seeking unifying principles and ignoring the rich diversity that biologists cherish. Proposed in 1997 by Geoffrey West, a physicist at the Santa Fe Institute, and his biologist colleagues Jim Brown and Brian Enquist, the theory has already explained some of biology's most far-reaching laws, mathematical regularities that *all* creatures seem to obey, from the smallest microbes up to elephants and whales.

For example, what are the daily energy needs of a mouse, a dog, or an elephant? To measure these needs, you can look at how much an animal eats, how much oxygen it uses, or how much heat it produces. However you make the measurement, the metabolic rates for diverse mammals always fall neatly on a curve when they're graphed versus their average body mass. Pound for pound, the little guys burn a lot more energy than we do. It's more revealing to plot *total* caloric intake versus mass (so now we're comparing different species in absolute terms, not on a pound-for-pound basis). When we use logarithmic axes, the data fall on a beautifully straight line, implying that the underlying law is a power law—just as we saw in fractals (in Lecture 16). The line rises with a slope of 3/4. Hence, bigger animals need to eat more than little ones, but only in proportion to their mass raised to the 3/4 power. When this law was reported by the veterinary scientist Max Kleiber in 1932, some biologists were upset (and some still are today). They had expected to find a 2/3-power law, based on simple considerations of surface area and its expected role in respiration, heat loss, and other physiological processes.

Kleiber's 3/4-power law has since been shown to hold over 27 orders of magnitude in body mass, extending from subcellular molecules to elephants. It testifies to the unity of life on Earth. It is one of the most comprehensive laws in all of science. What is the explanation for it? The mystery goes even deeper. There are many other scaling laws in biology, also with 1/4 powers of mass in them. The lifetime of a mammal increases in proportion to $M^{1/4}$, where M is its mass. The time intervals between heartbeats and breaths are also proportional to $M^{1/4}$. These results have the strange implication that a mammal of any size, from mouse to elephant, can expect to live for about 1.5 billion heartbeats or 300,000 breath cycles! Is this a coincidence or something fundamental about life? No one knows.

The theory of West et al. takes a step toward answering these questions by predicting the exponents in the scaling laws for metabolism, lifetimes, etc. They asked themselves: What is the *optimal* design of a network of branching tubes carrying nutrients, oxygen, or other resources, such that the resources are delivered to all parts of a 3-dimensional body as fast as possible and with minimum energy? Invoking some simplifying assumptions and the laws of fluid mechanics, they found that the network had to have a self-similar branching pattern. In other words, it had to be a fractal. Then they imposed the realistic constraint that the network's terminal units are invariant for creatures of different sizes. For the circulatory system, the terminal units are the capillaries (which are known to be roughly the same size for an elephant or a mouse). With these assumptions, the 3/4 power fell out of the math, as did the other known scaling laws. The same argument also explained the 3/4-power law seen at the cellular and subcellular level. The theory also correctly predicts that the size of the smallest possible mammal is a few grams—consistent with the mass of a shrew—and that mammalian cells in culture always have the same metabolic rate, regardless of what mammal they come from.

The optimality assumption rankles some biologists, since natural selection need not be optimal.

The theory is controversial. There are technical disputes about the calculation itself. The optimality assumption rankles some biologists, since natural selection need not be optimal. But it's intriguing that the observed behavior is close to optimal. In any case, optimality is a natural benchmark.

The modeling philosophy irritates some people. Drastic simplifications are used to expose broad trends—a common strategy in physics but anathema to biologists. Interesting details are missed. For example, humans live longer than the overall trend for mammals would predict, given our weight. West et al. would retort: That's the point! By highlighting the trend, one can tease out what requires biological explanation versus what follows more generically. ■

Essential Reading

For the most accessible introduction to the theory of West, Brown and Enquist, see: Whitfield, *In the Beat of a Heart*.

Supplementary Reading

Ball, *The Self-Made Tapestry*, chap. 5.

Liebovitch, *Fractals and Chaos Simplified for the Life Sciences*, 14–25.

McMahon and Bonner, *On Size and Life*.

Questions to Consider

1. What causes aging?

2. How much should you eat? Kleiber's law states that a warm-blooded animal of mass M, measured in kilograms, typically needs about $70M^{3/4}$ calories per day. Translated into pounds, this becomes $38W^{3/4}$ for an animal weighing W pounds. Apply this formula to yourself. (Hint: If you need help calculating the 3/4 power, multiply $W \times W \times W$ to get the third power (W^3), and then take the square root of that number, and the square root again, to get $\sqrt{\sqrt{W^3}} = \left((W^3)^{1/2}\right)^{1/2} = W^{3/4}$. Then multiply by 38 to estimate your daily caloric needs.)

Fractal Art

Lecture 18

This lecture examines two of the ways that fractals connect with art.

We begin with the computer-generated fractal images that adorned T-shirts, postcards, and coffee mugs in the 1980s. The most famous and dazzling of these is the Mandelbrot set. Mandelbrot was exploring the transition to chaos, as others had done earlier (see Lectures 8 to 11). But he studied simple iterations of points on a plane instead of a number line. The Mandelbrot set encodes a fantastic amount of information about the dynamics of these maps (much like the orbit diagram did for the logistic map; see Lecture 10). The most mild-mannered maps are shown as black points. The more explosive ones are color-coded according to how hot-tempered they are. The real pleasure comes when we zoom in for a closer look. Successive magnifications reveal phantasmagoric structures that look like sea horses and tendrils, lightning and kelp. Burrowing even deeper down, we are astonished to find the Mandelbrot set itself, reappearing in miniature, eerily repeating like variations on a theme. The pattern is infinitely rich, even though it was made by repeating a simple rule. It's also subtly self-similar—not blatantly and artificially so like the Koch curve (see Lecture 15) or other simple fractals. This combination of simplicity, richness, and subtle repetition is what makes these images so appealing. They're neither too regular nor too random—just like good art. These images captivated the public when they first came out around 1985. Personal computers had only just become available around 1980. Fractals were easy to program and compute, and they allowed many people—for the first time in their lives—to make their own mathematical discoveries and to feel how exhilarating math could be.

A more controversial application of fractals to art involves a recent analysis of the drip paintings of Jackson Pollock (1912–1956). Pollock famously said, "I am nature." His paintings are often described as "organic." He created them by channeling chaos, flinging and dripping paint from a stick onto huge canvases laid on the floor of his barn, all the while moving around, leaning off-balance in a controlled way. Are his paintings just random splatterings, as some cynics would say? Or are they fractals, full of hidden structure reflecting their chaotic genesis?

In 1999, a team led by physicist Richard Taylor reported that the drip paintings are fractals. They scanned photographs of the paintings and calculated their fractal dimensions, using the box-counting method we used for Norway's coastline (see Lecture 15). They found one fractal dimension for small scales (between 1 millimeter and around 2 or 3 centimeters). They attributed this to the chaos introduced by the fluid mechanics of the dripping process. They found a different dimension for the larger scales (between around 5 centimeters and 2.5 meters, the size of the canvas). This, they say, was determined by Pollock's technique and body movements.

Finally, they claim that as Pollock refined his technique over the years, the large-scale fractal dimension of his paintings increased. It started at around 1.1 in 1945, corresponding to a loose web of paint trajectories. Then it grew to around 1.5 by 1948, a value more like that of coastlines and other natural fractals. Finally, it reached 1.72 in a densely interlaced 1952 painting, *Blue Poles*, now valued at more than $30 million.

Pollock's alleged fractal signature was invoked as a damning line of evidence in a recent art authentication dispute. In 2005, Alex Matter, the son of a longtime friend of Pollock's, announced that he had found a stash of 32 drip paintings among his parents' belongings. If authentic, they would be worth many millions of dollars. Art historians had conflicting opinions about their authenticity. The Pollock-Krasner Foundation asked Taylor to analyze six of the paintings. He found none of them had Pollock's characteristic fractal geometry.

But in 2006, Taylor's methodology was ridiculed by another team of physicists. They showed that childish pictures of doodles would, when subjected to fractal analysis, give similar results to what Taylor found for Pollock's paintings. Taylor, Mandelbrot, and others dismissed this challenge, saying that the doodles are obviously not fractals, so computing their fractal dimension is meaningless. I agree with the critics that fractal dimension, by itself, is too blunt a tool to be convincing, either in art or science.

But I also believe that the paintings are indeed fractals, in part because of other, supporting lines of evidence. Taylor has produced his own drip paintings and demonstrated that they needn't be fractal. In failed fractals, small portions of the painting don't have the same general appearance as the whole. Thus, it was remarkable that Pollock somehow managed to make his drip paintings self-similar down to very small scales. This does *not* occur automatically. Apparently that's how Pollock instinctively knew when he was done painting, even if a casual observer couldn't see the logic.

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Taylor has also provided evidence that chaos was crucial to Pollock's process. He built a mechanical device he dubbed a "Pollockizer"—a pendulum that swings over a canvas while dripping paint onto it continuously. The pendulum normally swings regularly but can be made chaotic by perturbing it with a periodic sequence of electromagnetic kicks as it swings. The chaotic drip trajectories look like Pollock's. The non-chaotic ones don't. As of this writing, the jury is still out on the question of the new paintings' authenticity, although the evidence—fractal, chemical, and human expert opinion—weighs against them. ■

Essential Reading

Gleick, *Chaos*, 215–40.

Taylor, "Order in Pollock's Chaos."

Supplementary Reading

Peak and Frame, *Chaos Under Control*, chap. 7.

Peitgen and Richter, *The Beauty of Fractals*.

Rehmeyer, "Fractal or Fake."

Questions to Consider

1. Can images of the Mandelbrot set really be considered art? What about Pollock's drip paintings?

2. To what extent can artistic style be quantified mathematically? Will computers someday replace human connoisseurs in art authenticity disputes?

Embracing Chaos—From Tao to Space Travel

Lecture 19

The idea of chaos as a source of creativity also occurs in the writings of the philosopher Friedrich Nietzsche, who wrote, famously—you have probably heard this quote, but I like it—he said, "I say unto you, one must have chaos in oneself to be able to give birth to a dancing star. I say unto you, you still have chaos in yourselves."

In this lecture and the next, we'll take a look at how chaos can be used for practical purposes in engineering and technology. Artists and philosophers have long embraced chaos as a source of creative energy. In Lecture 18 we saw how Jackson Pollock channeled natural chaotic processes while creating his drip paintings. The idea of creative chaos also appears in the writings of philosopher Friedrich Nietzsche (1844–1900): "One must have chaos in oneself to be able to give birth to a dancing star." Perhaps the earliest expression of the fecundity of chaos is in the ancient Chinese philosophy of Taoism, from the writings of Lao Tzu in the 6th century B.C.E. Taoism is a philosophy based on nature, and Tao is the natural order of all things. Here, *hun-tun* (chaos) is the primal state, the mother of the universe. But unlike other conceptions of chaos (the abyss of the Greeks, the void of the Hebrews, or the malevolent disorder of Christian traditions), *hun-tun* signifies a primordial confusion that is fertile, pregnant with possibility, a source of life and creativity. In Taoism, nature is self-organizing. Order emerges spontaneously from chaos, with no need for a purposeful God to wrest order from disorder. There are parallels here with the modern scientific conception of chaos. Chaos contains hidden order, as we've seen in our discussions of strange attractors, iterated maps, orbit diagrams, and fractals. Nonlinear systems can organize themselves spontaneously (see Lecture 23).

Yet it's taken a long time for most scientists and engineers to see chaos as an ally. They traditionally avoided chaos, and where it was unavoidable, they squelched it. A new approach emerged around 1990. Close in spirit to Taoist teachings, the notion was to embrace chaos and to make use of its unique properties. An influential publication along these lines was titled "Controlling Chaos." By its very nature, chaos might seem uncontrollable. Yet in some ways the opposite is true. Precisely because tiny nudges to a chaotic system can have such potent effects, these systems are highly responsive. Edward Ott, Celso Grebogi, and James Yorke, theorists at the University of Maryland, showed mathematically that gently tapping on a chaotic system in the right way could coax it into periodic behavior. Rajarshi Roy, a physicist then at Georgia Tech, demonstrated this experimentally. He forced a laser to emit chaotically pulsating light and then nudged the chaos into various forms of periodicity. Operating in this way, the laser produced much higher power than usual, while remaining under control.

Aerospace engineers exploited a similar idea years ago. They purposely designed the F-16 jet fighter plane to be unstable, to enhance its maneuverability. The F-16 is almost impossible for a human pilot to control, but its computer automatically makes split-second corrections to keep the plane stable.

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Perhaps the most striking use of chaos has been in the design of fuel-efficient trajectories for space travel. Space flight is expensive. The costs of flying freight to the Moon are about $250,000 a pound. The conventional route is fast (3 days for a one-way trip), but it relies on a heavy-handed maneuver called a *Hohmann transfer*. A spacecraft parked in orbit around the Earth fires its engines until it reaches almost bullet speed, then coasts for 250,000 miles. Finally, to settle into lunar orbit, it pivots around and blasts its retro-rockets until it slows down enough for the Moon's gravity to capture it. The slow-down maneuver alone costs more than $130 million in fuel!

A new approach, invented by mathematician and former NASA scientist Edward Belbruno, exploits the chaos inherent in the three-body system of Earth, Moon, and spacecraft. He thinks of it as "surfing the gravitational field." Just as a big ocean wave carries a surfer along with it, Belbruno devised trajectories that "go with the flow" from the Earth to the Moon. The spacecraft *coasts* into lunar orbit without using any retro-rockets to slow down. There are only two catches:


NASA/JPL, Cici Koenig.

Interplanetary Superhighway Makes Space Travel Simpler.

- The trajectories become chaotically tangled when they reach a "fuzzy boundary," a gravitational tug-of-war region between the Moon and Earth.

- They take 2 years to get there!

Belbruno's superiors at NASA's Jet Propulsion Lab scoffed at this—and didn't trust chaos anyway—so they eventually let him go. But NASA came back asking for help in 1990. Could Belbruno salvage a Japanese space mission gone awry? Japan, aspiring to be the third country to reach the Moon, had launched two probes. One had headed toward the Moon but lost radio contact. Could Belbruno find a way to send the other spacecraft—a communications relay station the size of a desk—to the Moon, even though it was never intended for that purpose and had practically no fuel? The route had to carry the spacecraft to the Moon with just the right speed. Too fast and it'd sail past the Moon; too slow and it'd crash.

Belbruno (who was then collaborating with Jim Miller, the scientist from NASA who'd initially approached him) worked backward from the desired speed and altitude required. They found an amazing solution: a trajectory that first wandered a million miles away, out to where the Sun's gravity also became significant. This detour saved the day. Belbruno and Miller guided the spacecraft out to the Earth-Sun fuzzy boundary, then quickly pushed it through the chaotic tangle by firing the craft's rockets, after which it coasted back to the Earth-Moon fuzzy boundary. The Japanese spacecraft—named *Hiten*, meaning a Buddhist angel that dances in heaven—safely arrived at the Moon on October 2, 1991, after a 5-month trajectory.

Belbruno's ideas have now become mainstream. Extensions of his method, called *low-energy transfers* along the *Interplanetary Transport Network*, were used in NASA's unmanned *Genesis* mission and in the European Space Agency's *SMART-1*. Other ingenious uses of chaos may be crucial for the future of space travel, especially as NASA seeks inexpensive ways to achieve its mission of getting back to the Moon, and possibly to Mars by 2020. Even though low-energy transfers are too slow for people, they might be fine for delivering the supplies the astronauts would need on the Moon or Mars. ■

Essential Reading

Belbruno, *Fly Me to the Moon*.

Supplementary Reading

Frank, "Gravity's Rim."

Girardot, *Myth and Meaning in Early Taoism*.

Klarreich, "Navigating Celestial Currents."

Ott, Sauer, and Yorke, *Coping with Chaos*, chaps. 12–14.

Questions to Consider

1. When you're trying to be creative, do you prefer to make a mess first, or do you favor a more carefully controlled process? What are the pros and cons of each strategy?

2. Why might the ancient Chinese have arrived at such a different view of chaos from Western cultures?

Cloaking Messages with Chaos

Lecture 20

> **Today's digital technology makes it pretty tough to decrypt a cell phone call, but it wasn't always like that. … Do you remember when reporters intercepted Princess Diana's cell phone conversations with her lover, James Gilbey? It made the news because apparently his pet name for her was "Squidgy."**

This lecture continues our examination of the uses of chaos. Now the theme is encryption—how to use chaos to send secret messages. Our story begins around 1990, in the days before cell phone calls were routinely encrypted. Do you remember when reporters intercepted Princess Diana's cell phone conversations with her lover James Gilbey, later publicized as the embarrassing "Squidgy" tapes? Prince Charles's even more salacious conversations with Camilla Parker Bowles became public in the same way. When Newt Gingrich, then Speaker of the House, was under investigation for alleged ethics violations, his calls were intercepted by Democratic loyalists and used to humiliate him. In all these cases, even a minimal level of privacy would have helped—not military- or banking-level encryption, just enough to thwart a casual eavesdropper.

Chaos has two potential uses in this regard. First, it's noisy. If played through a loudspeaker, it sounds like static because it's made of all frequencies, just as white light is made of all colors. So it's good for concealing signals; it cloaks them over a wide band of frequencies. Thus, an eavesdropper might not even realize any information was being sent. And even if he did, he'd have trouble pulling the message out of the chaos. Second, chaos could act as a carrier (not just a concealer) of messages. Traditionally, signals are carried by sine waves (as in AM or FM radio). A sine wave has just one frequency. Chaos, being made of infinitely many frequencies, might be more versatile.

This line of thinking first occurred to Lou Pecora in 1988. Pecora is a playful, light-hearted physicist at the Naval Research Laboratory in Washington, D.C. His background is in solid-state physics, not communications or chaos theory. He became curious about chaos in the mid-1980s, when it was the hottest subject in physics. But to please his employer, he felt he needed to find a practical justification—military or otherwise—for devoting research time to chaos, then newfangled and trendy. As soon as he posed the question, he immediately thought of communications.

This raised the problem of synchronization. All communications require a transmitter and receiver to be synchronized. For instance, the process of tuning a radio to a particular station locks the receiver to the frequency of the broadcast transmission. Once synchrony is established, the song on the radio is extracted by *demodulating* it from the radio wave that carries it. So the first step was to figure out how to synchronize two chaotic systems. "Synchronized chaos" sounds like an oxymoron. The butterfly effect would seem to destroy any hope of achieving synchrony. That's true if the two chaotic systems are independent. But perhaps the butterfly could be dispatched by linking the systems somehow—they're communicating, after all! Pecora and his postdoctoral fellow Tom Carroll tried weeks of computer experiments on various pairs of chaotic systems, without success. One night, while rocking his baby daughter to sleep, Pecora had a brainstorm: "I need to drive chaos with chaos—I need to drive the receiver with a signal that comes from the same kind of system."

That idea worked. In computer simulations and then in experiments on electronic circuits, Pecora and Carroll found a way to synchronize two identical chaotic systems. The communication was one-way, with a chaotic transmitter driving a chaotic receiver. Pecora was asked to explain his work to the National Security Agency, which led to a scene reminiscent of a Monty Python sketch.

In 1993, graduate student Kevin Cuomo and his adviser Alan Oppenheim, electrical engineers at MIT, published the first experimental demonstration of chaotic encryption. Cuomo built a pair of matched circuits that ran the chaotic Lorenz system (see Lecture 7) and showed that it could transmit and mask human speech. When he demonstrated his circuits to my class, he masked a recording of the hit song "Emotions" by Mariah Carey. One student, apparently with different taste in music, asked, "Is that the signal or the noise?" Listening to the hiss of the encrypted music, one had no sense that there was a song buried underneath. Yet when the combined song plus mask was transmitted, the receiver synchronized almost perfectly to the chaotic mask, but not to the song. So after instant electronic subtraction, we heard Mariah Carey again.

The next breakthrough was a 1998 lab study conducted by Gregory VanWiggeren and Rajarshi Roy, physicists then working at the Georgia Institute of Technology. They extended chaotic encryption to lasers and fiber optics, instead of electronic circuits. Messages are converted into light and masked by the wild fluctuations of a chaotic laser. After transmission down the fiber, the optical message is decoded by a matched chaotic laser. This system transmits 150 million bits per second, thousands of times faster than electronic circuits.

Chaotic encryption has since been extended to real-world tests in the fiber-optic network of Athens, Greece, in 2005. This study was the first to demonstrate chaotic encryption outside of a lab. It worked with existing infrastructure, thus confirming its feasibility. It was fast and reliable. Transmission speeds reached 1 gigabyte of encrypted information per second—about the same as commercial transmissions—and lost only about 1 byte in every 10 million. Yet questions remain about the security of chaotic encryption. Kevin Short, a mathematician at the University of New Hampshire, has broken nearly every chaotic code proposed to date. Still, chaotic encryption might be useful as another layer of security in the e-commerce transactions that race around the world every day (which are currently encrypted by software using mathematically heavy algorithms). With the growing concerns about cyberterrorism, national security, and cell phone and Internet privacy, this potential application of chaos is worth exploring. ■

Chaotic encryption has since been extended to real-world tests in the fiber-optic network of Athens, Greece, in 2005. This study was the first to demonstrate chaotic encryption outside of a lab.

Essential Reading

Strogatz, *Sync*, chap. 7.

Supplementary Reading

Ditto and Pecora, "Mastering Chaos."

Ott, Sauer, and Yorke, *Coping with Chaos*, chap. 15.

Questions to Consider

1. If you were a code-breaker, what techniques would you use to extract a message buried in chaos?

2. How is the chaotic masking method different from other forms of encryption?

Chaos in Health and Disease
Lecture 21

Other researchers, as I mentioned, have their own promising algorithms, but nobody has fully solved the problem of epilepsy seizure prediction yet. But I think we're on the right track, and if we are, maybe Iasemidis's dream will come true. He dreams that some day an early warning system can be programmed on an implantable computer chip, planted in the brain, and then when the beast appears, the chip will trigger a kind of "brain defibrillator" to fire a preventive electrical volley or maybe release an anti-seizure drug.

Our next stop along the frontier of chaos research lies at the crossroads of mathematics and medicine. The topic is biological rhythms, from the electrical chatter of brain waves to the pulse of a beating heart. Building on decades of biological research about the function and mechanism of such rhythms, chaos theorists have been asking questions about their dynamics. Can the mathematics of chaos be used to predict epileptic seizures? The problem is important. Epilepsy afflicts 50 million people worldwide. During a seizure, an epileptic might convulse, black out, collapse, or stop breathing. Seizures attack suddenly, so epileptics dare not drive or even give their baby a bath. Just a few minutes of forewarning would help the patient get to a safe place and possibly take preventive medicine.

But is there any hope of predicting seizures? Clinicians traditionally viewed them as random, yet there do seem to be precursors. Some patients experience *auras*. They see strange lights, feel sick, or smell odors. Others are alerted by their dogs that a seizure is coming. Recently, bioengineer Leon Iasemidis and neurologist Chris Sackellares, and a few other teams of researchers worldwide, have been trying to predict seizures with the methods of chaos theory.

They seek clues in EEG (electroencephalogram) tracings taken before, during, and after a seizure. Such records come from patients who were being monitored in preparation for surgery to remove an epileptic focus from their brains. The data reveal that the EEG becomes *less* chaotic, not more, during a seizure. The EEG's fractal dimension and Lyapunov exponent (which measures the strength of the butterfly effect) both drop abruptly when a seizure starts.

Even more exciting, though, is what happens several minutes *before* the seizure. The time series of Lyapunov exponents measured from different electrodes start to synchronize, lining up in amplitude and phase. According to Iasemidis, this is "the mark of the beast," a clue that a seizure is imminent. His interpretation is that different brain regions are falling into pernicious lockstep.

What's still unknown is how often this pattern is followed by a seizure. In a preliminary study in 2003, Iasemidis found his algorithm correctly predicted 82% of seizures, with about 75 minutes of advance warning (on average). But it also made false predictions 3 or 4 times per day, which is about as many seizures as actually occurred. Several other researchers have their own promising prediction algorithms, but nobody has fully solved the problem yet. Iasemidis dreams that an early warning system can be programmed onto an implantable computer chip. When the beast appears, the chip will trigger a "brain defibrillator" to fire a preventive electrical volley or to release an anti-seizure drug.

Next we consider cardiac arrhythmias. Arrhythmias are irregularities of the heart's electrical activity. In the worst kind, *ventricular fibrillation*, the heart quivers and fails to pump blood, causing death within minutes if untreated. Sudden cardiac arrest kills over 300,000 Americans each year. Is arrhythmia a form of chaos? If so, can it be controlled with methods developed for other chaotic systems (see Lecture 19)? An elegant experiment along these lines was reported in 1992 by a team led by Alan Garfinkel, a physiologist and chaos theorist at UCLA.

They induced arrhythmia in rabbit heart tissue by applying drugs to it. The action of the drugs was gradual. The tissue beat periodically at first, and then progressed through two period-doubling bifurcations that later gave way to chaos. This is similar to the scenario we studied in the logistic map (in Lectures 9 and 10). The appearance of the interbeat interval plot confirmed that the arrhythmia was genuine deterministic chaos, not randomness.

Next the researchers controlled the chaos by harnessing the order hidden within it. They nudged the tissue into periodic behavior by stimulating it electrically, with sporadic but carefully timed pulses dictated by the theory. In contrast, simple pacing of the tissue didn't stop the arrhythmia.

What are the larger implications? Unfortunately, many arrhythmias are far more complex than this. In fibrillation, spiral waves of abnormal electrical excitation circulate on the heart, sometimes degenerating into multiple smaller wavelets. The point is that real arrhythmias are complex in both space *and* time. Many cardiologists and mathematicians are working hard to decipher their dynamics. With the insights that emerge, we may be able to develop gentler defibrillators, or perhaps more effective anti-arrhythmia drugs.

Finally, and most controversially, when chaos occurs in the human body, might it ever be a sign of health rather than sickness? This paradoxical idea was proposed by cardiologist Ary Goldberger of Beth Israel Hospital in Boston. In his lectures, he likes to show a graph of four patients' heart rate variability, and he asks the audience: Which of these is the sickest? The one with the most regular rhythm is! That person is suffering from congestive heart failure. The healthiest person has a surprisingly variable rhythm, with dance-like chaotic variations from beat to beat. The pattern is self-similar, at least in a statistical sense. It's a fractal process in time (see Lecture 16), displaying the same kind of variability at time scales of 300 minutes, 30 minutes, or 3 minutes. That variability reflects the normal workings of the involuntary nervous system. Goldberger believes that a bit of chaos is good for you. It allows you to adapt, to avoid rigidity in your physiology. He has also shown that variability decreases as we age. So stay young, stay chaotic! ■

Goldberger believes that a bit of chaos is good for you. It allows you to adapt, to avoid rigidity in your physiology.

Essential Reading

Gleick, *Chaos*, 275–300.

Supplementary Reading

Garfinkel et al., "Controlling Cardiac Chaos."

Glass and Mackey, *From Clocks to Chaos*, chaps. 1, 8–9.

Goldberger et al., "Chaos and Fractals in Human Physiology."

Iasemidis et al., "Adaptive Epileptic Seizure Prediction System."

Liebovitch, *Fractals and Chaos Simplified for the Life Sciences*, 225–35.

Winfree, *When Time Breaks Down*, chaps. 2, 5.

Questions to Consider

1. What diseases might be caused by too much chaos in the body? Which ones might be caused by too little?

2. In a similar spirit, name some diseases in which some part of the body becomes too rhythmic, or not rhythmic enough.

Quantum Chaos

Lecture 22

You can push one uncertainty down, but then the other one goes up. And if you push that one down, then the other one goes up. There's no way around it. It's built into the structure of the universe. There's no way to measure position and velocity simultaneously with unlimited precision. Why is that so terrible for chaos? Because it destroys our whole concept of state space.

This lecture discusses the blending of two great theories—chaos and quantum mechanics. They don't mix as easily as you might imagine. But once we find the right recipe, they're terrific together. Quantum chaos opens up all sorts of possibilities for future electronic devices, for making abstract art, and maybe for solving the greatest mystery in mathematics. Why aren't chaos and quantum theory kindred spirits? You might think they have a lot in common. They both highlight the unpredictability of nature and the limits of human knowledge. The frenzied motion of subatomic particles certainly seems "chaotic." But at a deeper level, the theories conflict.

First, their worldviews are diametrically opposed. Chaos is founded on determinism: The present uniquely determines the future. Whereas quantum theory speaks only of probabilities: Nothing is ever certain to happen. Second, chaos is mathematically forbidden in quantum mechanics. Quantum theory describes everything as a changing blur of probability waves. The differential equation that governs how these waves evolve (the *Schrödinger equation*) is linear. As such, it has no chaos in it. Only nonlinear equations can support chaos.

The third conflict is the worst. It involves Heisenberg's uncertainty principle. Heisenberg says we can't know the position and velocity of a particle simultaneously. But that's precisely the information we need to predict the motion of a pendulum or other classical mechanical system. So states and trajectories become blurry and ill-defined at the atomic scale. The fine structure required for chaos is smoothed out by quantum effects. So when we speak of quantum chaos, what we really mean are quantum signatures of deterministic chaos. What are the vestiges of chaos when we shrink a classically chaotic system down to atomic size? This question was first examined theoretically.

One model system was billiards. A particle (a billiard ball) moves in straight lines on a frictionless pool table, bouncing elastically whenever it hits the walls, and continuing like this forever. The question is: What are the trajectories like in the long run, regular or chaotic? The character of the motion turns out to depend on the shape of the table. If the table is rectangular or circular, the motion is regular. Two balls that start close together with similar velocities will stay together for a long time, diverging only slowly. But the motion becomes chaotic for a table shaped like a stadium. The trajectories diverge rapidly after a carom or two. This is the billiard version of the butterfly effect.

So far we've described billiards in a classical, deterministic way. What is the quantum counterpart? The trajectories of the billiard ball smear out to probability waves. A picture of these waves shows where the ball is most likely to be found. If the classical motion was regular, the waves look like the vibration patterns of a drumhead. The surprise comes when the classical motion is chaotic. Everyone expected the quantum picture would look speckled, like random waves colliding. But the actual picture has prominent "scars," as discovered by Eric Heller. Although such perfectly repeating orbits are very rare and unstable, they leave their mark by concentrating waves along them.

More recently, Heller has taken up a second career—as a quantum artist. His images of quantum waves and chaos are as beautiful as they are scientifically revealing. Some depict the paths of thousands of electrons, treated as classical particles, as they ski over a landscape with random bumps on it. This problem arises in nanoelectronics, in trying to understand what a 2-dimensional gas of electrons will do as it moves through a tiny semiconductor device called a *quantum dot*.

Quantum chaos may prove of practical value in the design of the nano-scale devices that will revolutionize the electronics of the future. At this scale, and at sufficiently low temperatures, electrons have to be treated as waves, not particles, and the principles of traditional electronics go out the window. Researchers have begun exploring this new realm by building quantum dots in the shapes of stadiums and circles. Now the billiard balls are electrons. At temperatures close to absolute zero, the electrons travel freely. They only suffer electrical resistance when they bounce off the walls of the device. The experiments show the telltale signs of quantum chaos. The electrical resistance of the device changes dramatically, depending on whether the classical motion is chaotic or regular.

The real shocker about quantum chaos, however, is that it links atoms to prime numbers, thus connecting the bedrock of reality to the most ethereal realm of human thought. For thousands of years, mathematicians have sought patterns in the prime numbers. Carl Friedrich Gauss discovered an approximate formula for number of primes less than a given number n. In the mid-1800s, Bernhard Riemann improved this formula by adding certain waves to it. The frequencies of those waves were based on a brilliant guess, now called the *Riemann hypothesis*, sometimes described poetically as "the music of the primes." Just as real music is made of sound waves, Riemann's waves are the building blocks of the primes. Proving Riemann's hypothesis is considered the greatest unsolved problem in math.

Thus, it seems there's a chaotic system—as yet undiscovered—whose quantum counterpart would unlock the secret of the primes.

What does this have to do with quantum chaos? Quantum systems have discrete energy levels, corresponding to waves vibrating at certain frequencies. Likewise, the primes are built from waves vibrating at a discrete set of frequencies, called *the zeros of the Riemann zeta function*. The shocker is the frequencies of the Riemann waves look uncannily like those for a quantum *chaotic* system, as pointed out by physicist Michael Berry. Thus, it seems there's a chaotic system—as yet undiscovered—whose quantum counterpart would unlock the secret of the primes. ■

Essential Reading

Rockmore, *Stalking the Riemann Hypothesis*, chap. 12.

Supplementary Reading

Berry, "Quantum Physics on the Edge of Chaos."

Du Sautoy, *The Music of the Primes*, chap. 11.

Gutzwiller, "Quantum Chaos."

Internet Resource

Explore the dynamics of billiards on tables of various shapes here: http://serendip.brynmawr.edu/chaos/home.html.

Eric Heller's online gallery of quantum art: www.ericjhellergallery.com.

Questions to Consider

1. Could chaos be the fundamental source of the randomness inherent in quantum mechanics?

2. Why do you think mathematicians are so obsessed with prime numbers? Explain why it makes sense to call these numbers the "atoms of arithmetic."

Synchronization

Lecture 23

You may have thought about this already. Women who are good friends who suddenly find that they are having their period around the same time as their best friend, month after month; their menstrual cycles have become synchronized. We now understand how that works. It's through some silent, chemical communication between women, mediated through pheromones.

It's surprising to learn how complicated the dynamics are of simple systems. In particular, all of these simple systems displayed chaos. At the frontiers of chaos research today, scientists have begun taking the natural next step. They've started exploring much larger dynamical systems with huge numbers of variables. We've devoted much of this course to dynamical systems having just a few variables. Examples include:

- The three-body problem (see Lecture 4).
- Lorenz's convection model (see Lectures 7 and 8).
- The one-variable logistic map (see Lectures 9 and 10).

You might expect the chaos to be exacerbated, and sometimes it is. On the other hand, sometimes these complex systems can behave very simply. Patterns emerge unexpectedly, and the system organizes itself. The individual parts of the system spontaneously cooperate.

In this lecture we'll discuss the simplest kind of self-organization, in which different parts of a system begin oscillating in unison. In other words, they *synchronize*. We've already discussed synchronized chaos in the context of encryption (see Lecture 20) and epilepsy (see Lecture 21). We're going to be talking about the synchronization of periodic things called *oscillators* rather than synchronization of chaos.

Rhythmic synchronization is important and pervasive. We tend to overlook it or think it trivial, perhaps because it comes so easily to us. Groups of people can march in step or keep a beat, even without a leader. Consequently, we have little intuition about what synchronization requires. Does it depend on human consciousness? No—animals can synchronize. Think of the graceful coordinated movements of flocks of birds, schools of fish, or fireflies that flash on and off in silent, hypnotic unison. These organisms may not be fully conscious, but still they can perceive what's happening around them. Maybe perception is necessary for synchronization? No. Thousands of mindless pacemaker cells in your heart synchronize their electrical rhythms automatically, triggering your heart to beat. Menstrual synchrony sometimes occurs among women who are close friends or roommates. The precise mechanism is unknown, but it seems to rely on unconscious chemical communication through odorless pheromones contained in sweat.

Even inanimate things can synchronize spontaneously. Christiaan Huygens serendipitously discovered this remarkable phenomenon in 1665, while observing a pair of pendulum clocks hanging from the same beam. Their swinging pendulums shook the beam imperceptibly, but enough to bring the clocks into lockstep. Any collection of oscillating systems—fireflies, metronomes, heart cells—can synchronize, if they are similar enough and if they interact in the right way. It doesn't matter whether the coupling occurs by chemicals, gravity, light, sound, vibrations, or electrical currents.

Rhythmic synchronization is important and pervasive. We tend to overlook it or think it trivial, perhaps because it comes so easily to us.

Working out the mathematics of collective synchronization has been a major thrust of nonlinear dynamics, running parallel to chaos theory. The pioneer was theoretical biologist Art Winfree. As an undergraduate at Cornell in 1965, he used a computer to simulate the dynamics of large populations of biological oscillators such as heart cells or fireflies. He assumed the oscillators were diverse, as expected for any biological population. In a thought experiment, Winfree imagined making the oscillators progressively more similar. He discovered that synchronization broke out suddenly, not gradually, as he made the population more uniform. A phase transition occurred at a certain threshold, somewhat like the sudden freezing of water into ice at a critical temperature.

Forty years later, Winfree's ideas shed light on a seemingly unrelated phenomenon: the spontaneous crowd synchronization that caused London's Millennium Bridge to wobble on its opening day. The Millennium Bridge is an elegant, minimalist footbridge across the Thames River. When thousands of Londoners streamed onto the bridge, it began to sway unexpectedly from side to side. Its designers were flabbergasted. The bridge was closed 2 days later and was repaired over the next 18 months at a cost of 5 million pounds and great embarrassment.

Investigations revealed that the swaying was caused by a strange cooperative effect. The pedestrians were unconsciously synchronizing their footfalls to the slight sideways movements of the bridge to keep their balance. In so doing, they inadvertently pumped energy into the bridge, making the vibrations worse. Video footage from the BBC showed the whole crowd rocking from side to side in unison.

This is not merely the old chestnut about soldiers having to break step while crossing a bridge. Soldiers arrive at a bridge already in sync. The pedestrians were not. Marching soldiers drive a bridge up and down. The pedestrians drove the Millennium Bridge sideways. Diagnostic experiments uncovered a threshold effect akin to Winfree's. The swaying occurred only if there were more than a critical number of people on the bridge. Below that number, the bridge was essentially motionless and the crowd remained desynchronized. The point is that civil engineers (and the bridge building code) were in the dark about the possible outbreak of synchronized dynamics that this complex system could display.

Fortunately, no one got hurt, and the bridge has now been stabilized. In 2005, my colleagues and I published a mathematical explanation of what happened on opening day. Our model combined ideas from chaos theory, Winfree's work on biological rhythms, and bridge mechanics. The analysis revealed that the wobbling and crowd synchronization were two aspects of a single instability process. The model also predicts the critical number of people for bridges with different properties, which may help in the design of future structures.

In the final lecture, we'll discuss why large dynamical systems are so important, not just for chaos theory but for the future of all of science. ■

Essential Reading

Strogatz, *Sync*, chaps. 1, 2, 4, 6.

Supplementary Reading

Pikovsky, Rosenblum, and Kurths, *Synchronization*, chaps. 1–5.

Strogatz et al., "Crowd Synchrony on the Millennium Bridge."

Winfree, *The Timing of Biological Clocks*.

Internet Resource

Play with a model for the synchronization of fireflies, using this applet: http://ccl.northwestern.edu/netlogo/models/Fireflies.

Questions to Consider

1. Describe some real-world situations in which synchronization (of people, companies, stock markets, and the like) is undesirable or even dangerous.

2. What are some possible adaptive explanations for synchronization among animals? For instance, why do birds flock? Why do crickets sometimes chirp in unison?

The Future of Science

Lecture 24

We've come to the end of our whirlwind tour of chaos theory, and it's time to reflect on the subject as a whole.

Let's begin with what should be the easiest question: What's the science of chaos all about? Our view of the subject has changed as we learned more and more. The same evolution happened historically, and it's still happening. In ordinary language, chaos is synonymous with disorder and utter confusion. But that's too broad for us. Such a definition would subsume chaos under probability theory, the study of randomness. That's wrong, because chaos theory deals exclusively with non-random, deterministic systems (where the present completely determines the future).

On the other hand, chaotic systems do have a disorderly aspect to them. They can impersonate random behavior because of their long-term unpredictability. That unpredictability is caused by the butterfly effect, the extreme sensitivity of a chaotic system's behavior to small changes in the initial conditions. So a second try would be: Chaos is the science of disorder in deterministic systems. And for a long time, it *was* that. That unpredictable side of chaos was dominant in Poincaré's work on the three-body problem (see Lecture 4). And also in Lorenz's work on his artificial weather model, through which he discovered the butterfly effect (see Lecture 5 and 6).

But soon, it came to be realized that there was also a tremendous amount of *order* in chaos. It was just order in many unfamiliar guises. Of course there was short-term order and predictability provided by the underlying differential equation. This would be true of any deterministic system, not just chaotic ones. But there was also unexpected long-term order in the way the system moved around in state space. This is the order embodied by strange attractors (see Lecture 7). And there was medium-term order in the way the system hopped along its attractor, when viewed under a strobe light. This is the order embodied by iterated maps (see Lecture 8).

Then we focused on iterated maps, instead of differential equations, as the dynamical systems of interest. This allowed us to uncover still more kinds of order (see Lectures 9 through 11). First, there was order as *parameters* were changed. (Recall, parameters are like knobs that can be manually controlled, like the growth rate of the insects modeled by the logistic map, or the setting of the brake on the waterwheel.) This order with respect to parameters was displayed vividly by the orbit diagram (see Lecture 10) and even more stunningly by the Mandelbrot set (see Lecture 18). Those diagrams were like Rosetta stones. By consulting them, we could see all the possible behavior of the system, in one icon. These diagrams showed where bifurcations occur—where the system dramatically changes its long-term behavior, from stable equilibrium, to oscillations, to chaos. Both of these Rosetta stones had incredible fractal structure, as did strange attractors themselves. And one particular aspect of this fractal order—the Feigenbaum numbers for the spacing between bifurcations leading to chaos—turned out to be *universal* for many chaotic systems, both real and mathematical (see Lectures 11 and 12). This led to the most stunning experimental confirmations of chaos theory, in experiments on fluids, circuits, chemical reactions, heart cells, lasers, and so on.

And the order in chaos soon got harnessed to practical advantage for all kinds of purposes, from encryption to space travel, and from quantum electronics to forecasting epileptic seizures.

Meanwhile, fractals took on a life of their own, as models of self-similar shapes or processes in the real world (see Lectures 13 to 18), not just in the abstract worlds of state space and orbit diagrams. And the order in chaos soon got harnessed to practical advantage for all kinds of purposes, from encryption to space travel, and from quantum electronics to forecasting epileptic seizures (see Lectures 19 to 22).

Behind all of this was a hidden puppeteer pulling the strings. The puppeteer was *nonlinearity*. None of our story would have been possible with linear systems—the ones studied by traditional science, in which causes are strictly proportional to effects and the whole equals the sum of the parts. So we've come to this definition: The science of chaos is about order and disorder in nonlinear, deterministic systems.

The real point of this definitional exercise, however, is that it shows us why chaos theory is going to be crucial to the science of the 21st century. Why? Because virtually all the major unsolved problems of science are fundamentally nonlinear. Consider the following:

- The orchestration of thousands of biochemical reactions in a single cell and their disruption when the cell turns cancerous.

- The battle between the immune system and the AIDS virus.

- The emergence of consciousness from the interplay of billions of neurons in the brain.

- The origin of life from a meshwork of chemical reactions in the primordial soup.

What makes these problems so vexing is their decentralized, dynamic character, in which enormous numbers of components keep changing their state from moment to moment, looping back on each other in ways that can't be studied by examining any one part in isolation. In such cases, the whole is surely not equal to the sum of the parts. These phenomena are nonlinear.

Unfortunately, our minds are bad at grasping nonlinear problems. We're accustomed to thinking in terms of centralized control, clear chains of command, and the straightforward logic of cause and effect. But in huge, interconnected systems, our standard ways of thinking fall apart. Verbal arguments are too feeble, too myopic. The good news is that, thanks to the development of chaos theory, we have now mastered the simplest nonlinear systems, the ones studied in this course. What awaits is a similar mastery of complex nonlinear systems.

By complex, I mean something with an enormous number of parts. These parts are typically diverse and interact with one another in bewildering networks. For example, consider the vast network of genes, proteins, and enzyme reactions that controls how every cell in your body grows and divides. One node in the network is p53, the most important tumor suppressor gene. It is known to be mutated (broken, basically) in about 50% of all cancers.

Mathematical models of the cell division cycle (for yeast, much simpler than the mammalian cell cycle, but still very complicated) are being developed by such pioneers as John Tyson and his colleagues at Virginia Tech. Their models are based on methods of chaos theory. Although such models are still avant-garde from the point of view of traditional cell biology, I believe they foreshadow the kinds of theoretical work that will prove indispensable in solving the riddle of cancer. After all, genes are nonlinear devices; stimulate them twice as strongly and they don't respond twice as much. The same is true of neurons. So all the mysteries of gene regulation and brain function—and all of biology for that matter—will ultimately require us to decipher complex nonlinear systems.

There are grounds for optimism. Even the most hard-boiled mainstream scientists now acknowledge that reductionism alone won't solve all the great mysteries we're facing. At every major research university, institutes are springing up with names like "functional genomics" and "systems biology." Biologists are teaming up with computer scientists and mathematicians to try to make sense of the dance of life at the molecular level. Chaos theory is playing a vital role in the new biology. Here's why:

- Although sequencing the human genome gave us an enormous list of parts (23,000 individual genes and the proteins they encode), we still have almost no clue how the interlocking activities of those genes and proteins are choreographed in the living cell, or how they go awry when cells turn cancerous.

- These are problems of dynamics.

- Traditionally, the genome has been viewed in more static terms, as a blueprint for the construction of proteins (the building blocks and molecular machines essential to life).

- But today we see that metaphor as too linear, a vestige of the assembly-line mentality of an earlier era.

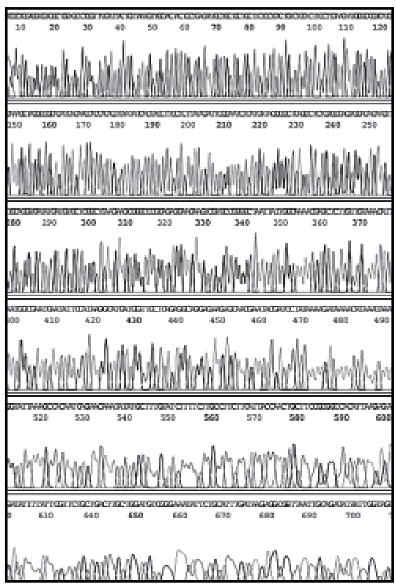
- Some of the most important genes (the so-called regulatory genes) code for proteins that alter the activity of other genes, turning them on or off, forming circuits and feedback loops.

- The genome starts to seem less like a blueprint and more like a computer.

- The functioning of this computer—and its malfunctioning when cells turn cancerous—will not be deciphered until we understand the dynamics of gene networks.

- Our era awaits its own Kepler, Galileo, and Newton. They found the laws of inanimate dynamics; we're still looking for the laws of life.

Fortunately, biologists at the cutting edge are now being trained in chaos theory. As part of their graduate education, systems biologists learn nonlinear dynamics to help them reverse-engineer the networks of life. Synthetic biologists are using chaos theory to design their own biological circuits, building the living versions of toggle switches, oscillators, and amplifiers out of genes and proteins instead of wires and transistors. Likewise, chaos theory is helping to shape other newfangled branches of modern science, from computational neuroscience to econophysics.


U.S. Department of Energy Human Genome Program,Genome Management Information System, Oak Ridge National Laboratory.

Genome Sequence Trace.

Chaos theory is also shedding light on perhaps the most puzzling question of all: consciousness. How can 100 billion neurons, each mindless on its own, fall in love or remember a favorite song or fret about its own mortality? Neuroscientists have found that when we recognize a face, or decide to pay attention to a conversation, far-flung parts of the cortex suddenly fire a volley of electrical signals, at 40 cycles a second, in synchrony. The synchronized oscillations emerge spontaneously out of chaos, the brain's baseline state. These studies paint a fascinating, though also disconcerting, picture of human existence. As we go about our lives, all that we feel and perceive and think may be just a reflection of the fleeting processes of synchrony in our brains. In every branch of science today, the tools, ideas, and interdisciplinary spirit of chaos theory are proving indispensable as scientists confront the greatest challenges of our times.

Let me end by saying how much fun it's been to teach this course. Thanks to The Teaching Company for giving me this wonderful opportunity. I hope you've enjoyed it too. And although trajectories are not allowed to cross in a deterministic system, I do hope our trajectories will cross again soon. Until then, be well and stay chaotic! ■

Essential Reading

Strogatz, *Sync*, chaps. 9–10, epilogue.

Supplementary Reading

Alon, *An Introduction to Systems Biology*.

Barabasi, *Linked*.

Buchanan, *Nexus*.

Questions to Consider

1. Having finished the course, what aspects of chaos theory did you find most surprising? Most appealing?

2. Select a story from the newspaper that is somehow related to the ideas in this course. In what ways do you see the issues differently now as a result of what you've learned?

Timeline

B.C.E.

c. 1000 .... Ancient peoples worldwide develop creation myths in which the universe arises from "chaos," a primeval state of emptiness or utter disorder.

c. 900–500 .... Chaos figures prominently in the majestic opening chapter of Genesis: "In the beginning God created the heaven and the earth. And the earth was without form, and void …"

c. 600–500 .... Lao Tzu, the founder of Taoism, writes, "There was something chaotic yet complete" before the universe was born, "I do not know its proper name but will call it Tao."

585 .... Thales correctly predicts a solar eclipse. Generally regarded as the founder of Greek science, he teaches that the world is orderly and ruled by comprehensible laws.

c. 520 .... Pythagoras discovers the laws of musical harmony and proves the Pythagorean theorem. He leads a mystical cult that holds that "all is number."

c. 300 .... Euclid writes the *Elements*, perhaps the greatest mathematical textbook of all time, and establishes a style of logical reasoning that dominates Western science for 2000 years.

C.E.

1621 .... Johannes Kepler publishes his three laws of planetary motion, thus solving an ancient mystery.

1638 .... Galileo publishes *Discourses and Mathematical Demonstrations Concerning the Two New Sciences*, which summarizes his discoveries about the laws of motion on Earth.

1666 .... Isaac Newton invents calculus and differential equations and discovers the law of universal gravitation. He is 24 years old.

1686 .... Newton finishes his masterpiece *Principia Mathematica*, which synthesizes and explains the laws found by Kepler and Galileo. This book launches modern science and gives rise to the concept of the universe as clockwork.

c. 1700–1900 .... Elaboration of Newton's clockwork universe. High points include the prediction and discovery of Neptune (1846) and the discovery of the laws of electricity and magnetism, thermodynamics, solid mechanics, and fluid mechanics.

1889.. While investigating the three-body problem of astronomy, Henri Poincaré discovers "chaos" (in the modern sense). Specifically, he finds that a deterministic system obeying Newton's laws can be unpredictable in the long run because of its extreme sensitivity to tiny changes in its initial conditions. The clockwork begins to creak, but few people notice.

1896.. Economist Vilfredo Pareto finds that the number of people whose personal incomes exceed a large value obeys a power-law distribution. Power laws later prove important in analyzing chaos and fractals.

1904 ....Mathematician Helge von Koch constructs an example of a curve that is continuous but has no tangent anywhere. Once viewed as monstrosities, such examples later seem natural when fractal geometry is developed.

1900–1960...................................... Two gigantic upheavals in physics—relativity and quantum mechanics—overshadow Poincaré's discovery of chaos. A few mathematicians keep the flame of Poincaré alive. They refine his geometric approach to differential equations, leading them to insights about chaos in iterated maps and classical mechanics.

1920–1960...................................... Applied mathematicians and electrical engineers use Poincaré's methods to analyze the nonlinear oscillators at the heart of radio, radar, transistors, and lasers.

1932.. Veterinary scientist Max Kleiber reports his 3/4-power scaling law of metabolism, later explained as a consequence of fractal branching networks in living things.

1943–1952.............................. Artist Jackson Pollock invents a new style of painting, deliberately channeling chaos and mirroring the fractals of nature.

c. 1950................................... Invention of the modern computer.

1954.. Geologists Beno Gutenberg and Charles Richter report a power law relating the frequency of earthquakes to the amount of destructive energy they release.

1960.. The meteorologist Edward Lorenz creates an idealized computer model of weather and discovers its sensitivity to initial conditions, now recognized as a signature of chaos. This suggests an explanation for why real weather is so hard to predict.

1960.. The mathematician Stephen Smale constructs the horseshoe map. It reveals stretching and folding to be the basic geometric mechanism behind chaos. Smale's rigorous math builds on earlier studies of chaotic differential equations by the mathematicians Mary

Cartwright, J. E. Littlewood, and Norman Levinson. Yet like their earlier work, Smale's contributions make no dent on the wider scientific community.

1963.. Lorenz uncovers order in chaos. He publishes the first image of a strange attractor, infers its infinitely layered structure, and extracts an iterated map that underlies the chaos.

1965.. Arthur Winfree, then a senior in engineering physics at Cornell, uses computer simulations to investigate synchronization in large systems of biological oscillators. He discovers that the onset of synchronization is analogous to a phase transition.

1971.. Mathematicians David Ruelle and Floris Takens publish an important article that gets scientists thinking about the longstanding mystery of turbulence in new ways. They also coin two buzzwords that soon catch on, when they suggest that "strange attractors" may underlie the "very complicated, irregular and chaotic" behavior that turbulence represents.

1972.. The "butterfly effect" enters the lexicon soon after Lorenz gives a lecture titled "Predictability: Does the Flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas?"

1975... Physicists Harry Swinney and Jerry Gollub test the predictions of Ruelle and Takens. They report experiments on turbulence in rotating fluids that contradict the conventional theory but seem consistent with the new approach.

1975... Mathematicians T. Y. Li and James Yorke publish "Period Three Implies Chaos." "Chaos" now becomes the nickname of an entire intellectual movement.

1975... Benoit Mandelbrot publishes a book in French, *Les objets fractals: Forme, hazard et dimension*, in which he introduces fractal geometry, aimed at quantifying all that is fragmented and irregular.

1976... Theoretical biologist Robert May publishes a review article in *Nature* in which he makes an "evangelical plea" for educators to start teaching students about "the wild things that simple nonlinear equations can do." This article introduces chaos and the logistic map to a broad community of scientists.

1976... Astronomer Michel Hénon reveals, in dramatically clear computer graphics, the infinitely layered, self-similar microstructure of a strange attractor. In so doing, he helps others to grasp what Lorenz had meant back in 1963, when he described the attractor as an "infinite complex of surfaces."

1978.. After several rejections, physicist Mitchell Feigenbaum manages to publish his ideas about universality in the period-doubling route to chaos.

1980–1982...................................... Experiments on fluids, electronic circuits, lasers, chemical reactions, and heart cells confirm Feigenbaum's predictions.

1984.. Physicist Eric Heller predicts "scars" in the wave functions of certain quantum systems whose classical counterparts are chaotic.

1985.. The Mandelbrot set appears on the cover of *Scientific American*. The lay public becomes enchanted by fractals, which soon appear on T-shirts, posters, and screensavers.

1986.. Physicist Michael Berry links quantum chaos to the Riemann zeta function, which controls the enigmatic distribution of prime numbers.

1986.. Sir James Lighthill publishes a remarkable collective apology on behalf of all scientists for misleading the public about the unpredictability lurking in Newton's laws.

1987.. James Gleick's book *Chaos* becomes a bestseller.

1990.. Chaos theory stars in Michael Crichton's novel *Jurassic Park*.

1990... A new trend begins: Having understood chaos, scientists try to use it for practical purposes.

1990... NASA scientists Edward Belbruno and Jim Miller devise a way to get an unmanned Japanese space probe to the Moon using hardly any fuel, by exploiting chaotic trajectories through the solar system.

1990... Ed Ott, Celso Grebogi and Jim Yorke publish an influential paper, "Controlling Chaos."

1990... Physicists Louis Pecora and Tom Carroll show that two identical chaotic systems can be synchronized, opening the door to practical applications in communications.

1992... Using chaos-control techniques, physiologist Alan Garfinkel and collaborators show that a form of cardiac arrhythmia in lab animals can be stopped by gentle, precisely timed electrical stimuli.

1993... Electrical engineers Kevin Cuomo and Alan Oppenheim experimentally demonstrate the use of chaos for enhancing the privacy of electronic communications.

1993... Steven Spielberg brings chaos theory to the masses with his blockbuster movie version of *Jurassic Park*.

1993 ....Tom Stoppard brings chaos theory to the theatre-going elite with his play *Arcadia*.

1994.. Mathematician Kevin Short decodes communications masked by chaos, raising doubts about the security of this encryption method.

1996.. Bioengineer Leon Iasemidis and neurologist Chris Sackellares propose methods based on chaos theory to predict epileptic seizures.

1997.. Fractal theory proposed by Geoffrey West, Jim Brown, and Brian Enquist explains Kleiber's law of metabolism (1932) and scaling laws for other biological processes.

1998.. Applied mathematicians Duncan Watts and Steven Strogatz report that the small-world phenomenon is a generic feature of networks in nature, society, and technology. Networks become the next big topic in chaos and complexity research.

1998 ....Laser physicists Gregory VanWiggeren and Rajarshi Roy experimentally demonstrate the use of synchronized chaotic lasers for encryption of high-speed data transmission in optical fibers.

1999.. Fractal analysis of Jackson Pollock's drip paintings reported by physicist Richard Taylor and colleagues.

1999.. Power laws proposed as another organizing principle for complex networks, by physicists Albert-Laszlo Barabasi and Reka Albert.

2000.. Human genome sequenced, in “first draft” form.

2000 ....London’s Millennium Bridge wobbles unexpectedly on opening day when pedestrians spontaneously synchronize their footfalls, exemplifying the surprises lurking in complex nonlinear systems.

2001.. The NASA Genesis mission uses the interplanetary superhighway, a freeway through the solar system uncovered by chaos theory.

2005.. First successful implementation of chaotic encryption outside the lab, demonstrated in the fiber-optic network of Athens, Greece.

Glossary

accumulation point: The limit of a converging sequence of numbers. For example, the sequence 1, 1/2, 1/4, 1/8, 1/16, … has an accumulation point at 0, because the sequence approaches (or "accumulates" at) 0 as a limit. In the context of the period-doubling route to chaos, the accumulation point refers to the parameter value where the fig tree becomes infinitely bifurcated and chaos ensues. (For the logistic map, this occurs when the parameter is equal to 3.5699... .) Hence the accumulation point also marks the onset of chaos.

aperiodic: Changing without ever repeating or settling down to equilibrium.

attractor: A state, or collection of states, that represents a system's natural, long-term behavior. If a small disturbance nudges the system away from these states, the system quickly relaxes back onto them, as if "attracted" by them. There are three main kinds of attractor: (1) an attracting fixed point, which represents a stable equilibrium state; (2) an attracting cycle, which represents a stable periodic behavior; and (3) a strange attractor, which represents a permanently self-sustaining form of chaos.

bifurcation: A sudden, qualitative change in a system's long-term behavior caused by changing one of its parameters.

butterfly effect: The phenomenon in which a small change in a chaotic system's initial state (metaphorically caused by the flap of a butterfly's wings) leads to a huge change in what follows, compared to what would have happened otherwise.

chaos: (1) Colloquially: utter confusion. (2) Theologically: the primordial state before creation. (3) Scientifically: seemingly random, unpredictable behavior in a system that is nevertheless governed by deterministic laws. Chaos in this last sense is a subtle mix of order and randomness; it is predictable in the short run (because of determinism) but unpredictable in the long run (because of sensitivity to initial conditions).

chaos theory: The interdisciplinary study of nonlinear dynamical systems and their applications. Broader than the study of chaos per se, in the sense that it includes such non-chaotic phenomena as oscillations, waves, and patterns. Broader also in that it is not confined to mathematics; chaos theory encompasses all the scientific and technological areas in which nonlinear phenomena arise.

clockwork universe: A worldview in which everything in the universe unfolds according to Newton's laws, with no room for chance or free will.

complex system: A nonlinear dynamical system composed of an enormous number of interacting parts.

convection: Rotating flow of a fluid, such as a large mass of air or water, driven by differences in its temperature from place to place.

cycle: Anything that repeats. Represented by closed orbits in state space (for a system governed by a differential equation) or by a discrete set of points (for a system governed by an iterated map).

deterministic system: A system in which the present uniquely determines the future. Later states evolve from earlier ones according to a fixed law; only one thing can happen next.

differential equation: An equation that dictates how the state of a system changes in the next instant, given its current state. More generally, any equation that relates a system's variables to its derivatives (its rates of change with respect to time or space). Traditionally, one tried to solve such equations by using formula-based, analytical methods that rely on calculus. This approach requires great ingenuity and works best when the equation is linear or can be recast as linear. But for nonlinear differential equations, formulas are typically impossible to obtain, so one needs to resort to qualitative methods (as pioneered by Poincaré) or computational methods (as illustrated by the work of Lorenz).

dimension (of a fractal): Any of a set of related numbers that quantifies the degree of irregularity, jaggedness, or roughness of a fractal and gives information about its scaling and self-similarity properties. Reduces to the commonsense notion of dimension for Euclidean shapes: A line or curve is one-dimensional, a plane or surface is 2-dimensional, etc. Fractals have higher dimensions than their Euclidean counterparts, e.g., a fractal curve has dimension somewhere between 1 and 2; higher dimension means more convolution.

dimension (of a state space): The number of axes, one for each state variable. For example, the state of a pendulum is defined by two variables: its current angle and velocity. Hence its state space is 2-dimensional.

double pendulum: A toy used to demonstrate chaos. A lower pendulum hangs from an upper one. If given enough initial energy, the system behaves crazily, with the lower pendulum sometimes swinging and sometimes whirling, all unpredictably.

dynamical system: Any system that changes over time according to a fixed rule. There are two main types: differential equations (in which time is regarded as flowing continuously) and iterated maps (in which time is regarded as advancing in discrete steps).

equilibrium: An unchanging state. Note that nothing is implied about stability. An equilibrium could either be stable (a ball resting at the bottom of a bowl) or unstable (a pencil balancing on its point).

Euclidean: Relating to Euclid's classical geometry of lines, circles, planes, cones, etc. Smooth, as opposed to fractal.

exponent: The power p in an expression like x^p.

exponential (growth or decay): A rapid form of growth or decay, in which a variable changes by a constant *factor* over equal intervals of time. Much faster than linear growth or decay, in which a variable changes by a constant *amount* over equal intervals of time.

fig tree: The tree-like portion of the orbit diagram for any one-humped map (such as the logistic map or the sine map) that displays an infinite sequence of period-doubling bifurcations as a parameter is varied. Whimsically named in honor of Mitchell Feigenbaum (*Feigenbaum* = "fig tree" in German), who elucidated its universal properties. He showed that each wishbone in the tree has a width (i.e., distance along the parameter direction) that is always about 4.7 times narrower than the wishbone to the left of it and a height (i.e., distance along the state of the system) that is always about 2.5 times shorter. Close to the onset of chaos, these shrinkage factors universally converge to very specific, very peculiar numbers: 4.6692... and 2.5029... . These are fundamental constants of chaos, as basic to period doubling as pi is to circles.

fixed point: The state-space representation of equilibrium; an unchanging state.

fractal: An object, shape, or temporal process in which small parts resemble the whole. The resemblance may be exact in certain mathematical cases, but in most real-world cases it is only approximate or statistical. Furthermore, it may hold only over some limited range of length or time scales.

fuzzy boundary: A complicated region between the Earth and the Moon (or between two other celestial bodies) in which a spacecraft would be chaotically balanced, like a surfer riding the crest of a wave, pulled by the competing gravity of both bodies and pushed by centrifugal forces created by its own orbit.

growth rate parameter: The parameter r in the logistic map that controls how fast the population grows and hence regulates its dynamics. Sometimes called the **steepness parameter**.

heavy tail: The tail of a power-law probability distribution. "Heavy" in the sense that extremely large events occur much more often than they would in a normal distribution (standard bell curve). Mathematically, heavy tails decay slowly (in inverse proportion to the size of the event, raised to some positive power), rather than exponentially fast.

Hénon map: A pedagogical example, devised by the astronomer Michel Hénon, for exploring chaos and the fractal structure of strange attractors. Defined mathematically as a map involving two variables x_n, y_n, updated according to the rule $x_{n+1} = y_n + 1 - ax_n^2$, $y_{n+1} = bx_n$. Following Hénon, the parameters are usually taken to be $a = 1.4$, $b = 0.3$, because this produces a strange attractor whose internal structure is plainly visible.

horizon of predictability: Roughly speaking, the time up to which a chaotic system's behavior can be successfully predicted despite some uncertainty in its initial state. For times much longer than this, accurate prediction becomes impossible. Mathematically, the horizon of predictability is defined as the reciprocal of the system's Lyapunov exponent.

hump-shaped: Resembling an arch or upside-down parabola.

infinite complex of surfaces: Lorenz's description of the multi-sheeted structure of the strange attractor he discovered.

initial conditions: The state of a system at the beginning of an experiment, observation, or any other stretch of time that may be of interest to an investigator.

iterated map: A mathematical operation in which a number is fed into a function (a "map" or "mapping") to generate a new number. This new number is then fed back into the map to generate another new number, and the process is repeated, or "iterated," indefinitely.

Kepler's laws: The three laws of planetary motion discovered by the astronomer Johannes Kepler: (1) The planets move in elliptical orbits around the sun, with the sun at one focus of the ellipse. (2) An imaginary line joining the planet to the sun sweeps out equal areas in equal times. (3) The ratio T^2 / a^3 is the same for all planets, where T is the time required for one orbit, and a is the average of the planet's longest and shortest distances from the Sun.

Kleiber's law: An empirical regularity linking the average basal metabolic rates of all mammals—namely, a mammal of mass M (in kilograms) needs to eat about $70M^{3/4}$ calories per day. First reported by the veterinary scientist Max Kleiber in 1932, in a study spanning rats to steers. The law is for basal metabolism, assuming no special exertion (you can change your law by running or swimming vigorously each day).

Koch curve: A mathematical fractal formed by starting with a line segment, then repeatedly replacing its middle third with the other two legs of an equilateral triangle. Constructed by Swedish mathematician Helge von Koch in 1904.

linear system: A system in which alterations in the initial state will result in proportional alterations in any subsequent state. More loosely, a system in which causes are proportional to effects, and the whole is equal to the sum of the parts.

logistic map: A canonical model of the transition from order to chaos, originally inspired by studies of population growth. Mathematically defined as the iterated map $x_{n+1} = rx_n(1-x_n)$. Here x_n is a real number between 0 and 1, representing the state of the system at discrete times $n = 0,1,2,\ldots$, and r is a parameter, usually taken to lie between 0 and 4. In its original biological context, x_n is proportional to the population size of generation *n*, and r is the per capita growth rate.

long tail: See **heavy tail**.

long-term behavior: The behavior of a dynamical system after it has settled onto its attractor and "forgotten" the artifacts introduced by the particular way it was started.

Lorenz attractor: The butterfly-shaped strange attractor of the Lorenz system.

Lorenz map: An iterated map defined by the consecutive maximum values of the variable z in the Lorenz system. The map takes one maximum to the next.

Lorenz system: A system of three nonlinear differential equations studied by meteorologist Edward Lorenz, in connection with a simplified model of fluid convection. Can be visualized intuitively using its mechanical analogue, a chaotic waterwheel that erratically changes its direction of rotation. Famous for being the first system in which strange attractors were discovered. (For those with a math background: The Lorenz equations are given by the differential equations $x' = \sigma(y-x)$, $y' = rx - y - xz$, and $z' = xy - bz$, where prime denotes time derivative. Here $\sigma, r, b > 0$ are parameters, usually taken as $\sigma = 10$, $r = 28$, and $b = 8/3$, as in Lorenz's 1963 paper.)

Lyapunov exponent: The rate at which nearby trajectories diverge in a chaotic system. Defined mathematically as λ, the rate of separation, in the formula $d(t) \approx d_0 \exp(\lambda t)$, where $d(t)$ is the exponentially growing distance between the trajectories, and t is the time for which they've been separating.

Mandelbrot set: An amazingly complex and beautiful fractal set associated with a certain family of 2-dimensional maps of the plane to itself. (For those with a math background: The Mandelbrot set is defined as follows. Start with a complex number c, and generate the sequence c, $c^2 + c$, $(c^2 + c)^2 + c$, and so on; the next number is always the previous number squared plus the original c. Iterate indefinitely. The question is: Does this sequence head off to infinity or not? If any member of the sequence ever lies outside a certain square [namely, the square region of complex numbers having both real and imaginary parts between 2 and −2], the sequence is provably heading off to infinity. In this case, the point c is considered outside the Mandelbrot set, and c is color-coded by how many iterates it requires to escape from the square. Otherwise, the sequence remains in the square forever, and the point c is colored black, indicating it is in the Mandelbrot set.)

map: A function; a mapping; a transformation (all these terms are synonymous). Roughly speaking, a map eats states and spits them out, transformed. Think of a map as a machine that takes a state as input and "maps" it to a new state as output, using a given mathematical rule.

nonlinear: A system that is not linear; alterations in the initial state need not produce proportional alterations in subsequent states. More loosely, a system where the whole can be more or less than the sum of the parts, and causes can generate surprisingly disproportionate effects. All chaotic systems are nonlinear, but nonlinear systems comprise a much larger universe than just the chaotic ones. In that sense, "chaotic" is a subset of "nonlinear."

non-periodic: See **aperiodic**.

onset of chaos: Value of a parameter at which periodic or otherwise regular behavior ceases and chaos begins.

orbit: A geometric representation of the successive states of a dynamical system as it changes from moment to moment. Appears as a smooth curve in state space (for a differential equation), or a discrete set of points, each one hopping to the next (for an iterated map). Called an orbit by analogy with the orbit of a planet as it glides through outer space; the difference is that here the gliding occurs in an abstract mathematical state space. Synonymous with **trajectory**.

orbit diagram: A diagram showing how the attractor of a system changes as a parameter is varied. Called an orbit diagram because it depicts the long-term orbit (the attractor) of a system as a function of one of its parameters.

oscillator: Any dynamical system that cycles repetitively through a set of states.

parameter: A feature of a system that stays constant as time progresses. The natural evolution of the system does not change any of its parameters; only its variables will change. (For instance, when a pendulum swings back and forth, its angle changes, but its mass and length do not. Thus, by this definition, the mass and length of a pendulum are parameters, whereas the angle is a variable.) However, it is often possible for an experimenter to intervene and manually change a parameter. The most useful parameters are those that an experimenter can adjust, as if by turning a knob—especially if those adjustments cause the system to behave in new, interesting ways.

period: The time required for one complete cycle of an oscillation.

period doubling: A bifurcation seen in certain systems, in which a periodic attractor abruptly changes to another form and now takes twice as long to complete a cycle.

period-doubling route to chaos: An infinite sequence of periodic-doubling bifurcations as a parameter is varied, culminating in chaos. Seen in the logistic map and in many real systems investigated experimentally. The documentation of its predicted universal features provided one of the great triumphs of chaos theory.

periodic: Repeating exactly; cyclic.

periodic window: A continuous set of parameter values within which a system shows stable periodic behavior. Typically bracketed between parameter values for which the behavior is chaotic.

phase transition: A bifurcation in the state of a complex system, in which the overall character of the system changes radically, altering enormous numbers of components at once. A classical example is trillions of water molecules freezing into a solid block of ice when the temperature falls below the freezing point. Spontaneous synchronization of flashing fireflies or other oscillators is an analogous phenomenon, except it represents an alignment of rhythms in time, not molecules in space.

power law: An algebraic relationship in which one variable is proportional (or inversely proportional) to another variable raised to some power. For example, Newton's inverse square law of gravity is a power law because it relates the gravitational force between two bodies to the inverse second power of the distance between them. Kleiber's law relates basal metabolism to the 3/4 power of a mammal's body mass. Power laws are often associated with fractals, but not always.

probability distribution: A mathematical function (often graphed as a bell curve or the like) that shows the relative likelihood of a random variable taking on a certain value. For example, human heights lie on a bell-shaped probability distribution centered somewhere between 5 and 6 feet tall. Probability distributions are also useful for showing the likelihood of possible states for a chaotic system because some features of chaotic systems resemble those of truly random ones.

quantum chaos: A misnomer; not chaos in quantum systems. Rather, the telltale sign in a quantum system that chaos lurks in its classical Newtonian counterpart. Sometimes called "quantum signatures of classical chaos" for this reason.

random system: A system in which the progression from one state to another is not uniquely determined by any law; a system that is not deterministic. Loosely speaking, anything that can happen, can happen *next*.

reductionism: A philosophy of science that holds that all complex phenomena can (and should) be understood by "reducing" or breaking them into simpler pieces.

route to chaos: A series of bifurcations, starting from equilibrium and periodic states and ending in chaos, as a parameter is varied.

scale: The typical size of the main features of an object, shape, or process. Somewhat subjective; depends on which features are of interest in a given setting. For example, the scale of vortices in the atmosphere is hundreds of miles if we're interested in hurricanes, or a few feet if we're interested in piles of swirling leaves.

scale-free: A shape or process that displays similar behavior over such a wide range of scales that it is effectively independent or "free" of any one scale. Often used in reference to fractals or other systems showing power-law behavior.

scaling law: An algebraic relationship, often a power law, that describes how the measured value of some property depends on the resolution or scale (the length of the "yardstick") used to make the measurement.

scar: A region of high probability in a quantum system. Caused by probability waves adding up constructively along a closed orbit of the classical version of the system. The name is intended to convey the idea of a remnant: Just as a real scar is the remnant of a wound, a quantum scar is the remnant of a classical closed orbit, still visible even after descent into the quantum regime.

self-similar: Composed of parts that, when magnified, resemble the whole. In idealized mathematical fractals, the resemblance is exact and extends down to arbitrarily small parts; in real fractals, the resemblance is approximate and holds over a limited range.

sensitivity to initial conditions: The defining property of chaos; initially small disturbances to the state of a system grow exponentially fast as time progresses.

stable: A fixed point or cycle is stable if trajectories that start close to it remain close to it forever. In physical terms, a state is stable if small disturbances to it stay small. In many cases, these small disturbances eventually die out. Not synonymous with equilibrium; an equilibrium may be stable or unstable.

state: The condition of a system at one instant; the totality of information that (along with the governing equation of motion) is needed to predict what the system will do next.

state space: A geometric representation of all possible states of a system, it has as many dimensions as the number of variables needed to specify the system's state. The state space of a pendulum has two axes, one for its angle and one for its velocity. The state space for the most general three-body problem of astronomy has 18 dimensions, since we must specify the x, y, and z coordinates of each particle as well as its velocities along those axes. Thus each particle requires 6 numbers, and there are 3 particles, hence a state space of $6 \times 3 = 18$ dimensions.

steepness parameter: The parameter in the logistic map that controls how steep it is and hence regulates its dynamics. Also called the **growth rate parameter**.

strange attractor: An attractor with a fractal structure. Called "strange" because of its fractal geometry; all attractors studied previously were points, circles, or other smooth shapes in state space.

synchronization: A coordinated form of motion in which parts of a system move in unison or at the same frequency.

system: Any entity that can change over time.

tail: The far end of a probability distribution, corresponding to outliers or extreme events (large earthquakes, enormous incomes, etc.).

thermometer of complexity: A metaphor to illustrate what is known and unknown about dynamical systems.

three-body problem: The problem of determining the motion of three bodies pulling on each other by gravity. Now known to be unsolvable, except by brute-force computer simulation. Analysis of it led Poincaré to the discovery of chaos.

trajectory: The sequence of states followed by a dynamical system as time progresses. Synonymous with **orbit**.

transient behavior: Short-term behavior seen in a dynamical system before it reaches its attractor.

turbulence: Irregular motion of a fluid, such as air or water, in which the velocity at a given point varies erratically in magnitude and direction as time progresses.

two-body problem: The problem of determining the motion of two bodies pulling on each other by gravity. Solved by Newton, and used by him to explain Kepler's laws of planetary motion.

universal: Independent of the specific details of a mathematical or physical system.

unstable: Not stable. Small deviations from an unstable state grow over time.

U-sequence: The sequence in which stable cycles occur for the logistic map, or any other hump-shaped iterated map, as its steepness parameter is increased. Called "U" for "universal"; the same sequence occurs for any map with the same overall shape, independent of its precise algebraic form. The U-sequence, up to period 6, is 1, 2, 4, 6, 5, 3, 6, 5, 6, 4, 6, 5, 6. The U-sequence up to period 5 means the 6s drop out, leaving 1, 2, 4, 5, 3, 5, 4, 5. Notice that any definition of the U-sequence also requires (as part of the definition) an arbitrary upper limit on the cycle periods being considered. Otherwise, the period-doubling sequence at the beginning causes trouble; accounting for it alone requires listing infinitely many numbers 1, 2, 4, 8, 16, 32, ... , and you would never get to list the other, subsequent periods like 6, 5, 3, etc., that occur farther to the right in the orbit diagram. Also, once you allow any higher period as the upper limit, that new period will insert itself in places throughout the sequence in various complicated ways.

variable: A feature of a system that changes as time progresses.

vector field (on state space): A rule that assigns a vector to each point in state space, indicating the direction and speed with which one state changes to the next; the geometric counterpart of a differential equation.

waterwheel, chaotic: A mechanical analogue of the Lorenz system, used in lab demonstrations of chaos. Devised by applied mathematicians Willem Malkus and Lou Howard, then at MIT. Shows irregular reversals, turning clockwise and then counterclockwise after an unpredictable number of rotations each time.

window, periodic: See **periodic window**.

Biographical Notes

Belbruno, Edward (b. 1951). American mathematician with expertise in celestial mechanics and dynamical systems. An artistic soul and a bit of a maverick. While working for NASA's Jet Propulsion Laboratory, he used chaos theory to devise a new strategy for getting a spacecraft to the Moon at extremely low cost in fuel. Rather than fighting the gravitational tugs of the Earth and Moon, the spacecraft goes with the flow. It glides along chaotic trajectories created by the delicately balanced pulls of the Earth, Moon, Sun, and centrifugal forces. The savings in fuel are tremendous, but the journey is slow and so is suitable only for unmanned spacecraft. This method was used to rescue the Japanese space probe *Hiten* in 1990.

Feigenbaum, Mitchell (b. 1944). American theoretical physicist. Looks the part of a genius, with the brow and swept-back hair of a Romantic composer. While working at Los Alamos in the mid-1970s, he used a pocket calculator to uncover a stunning kind of universality in chaos. Roughly speaking, he found that diverse systems go chaotic in the same way. In particular, systems that follow a period-doubling route to chaos obey certain universal scaling laws, and always with the same scaling exponents. He explained how such universality arises by borrowing a Nobel Prize–winning technique (the "renormalization group") from statistical physics. His predictions were confirmed in controlled experiments on chick heart cells, electronic circuits, chemical reactions, and convection in liquid mercury.

Heller, Eric (b. 1946). American computational physicist. Predicted "scars" (regions where quantum waves add up coherently) as a signature of quantum chaos. These are produced by unstable periodic orbits in the chaotic classical counterpart of the quantum system. Heller has generated gorgeous computer graphics of quantum and chaotic systems, leading him to a second life as a quantum artist.

Hénon, Michel (b. 1931). French mathematician and dynamical astronomer. Invented a 2-dimensional chaotic mapping, now known as the Hénon map, that rendered the infinitely layered microstructure of strange attractors plainly visible for the first time.

Lorenz, Edward (1917–2008). American mathematician and dynamical meteorologist. Modest and prone to speaking in a soft monotone, he gave the impression of a laconic Yankee farmer, not that of one of the world's greatest scientists (which he was). Around 1960, he concocted a simple computer model of artificial weather, and in so doing discovered the butterfly effect in chaotic systems. He later simplified this model to a system of three innocent-looking differential equations, known now as the Lorenz system. This "little model" (as he called it) launched the chaos revolution. Unlike the chaos found by Poincaré in the three-body problem, which was transient and therefore hard to visualize, Lorenz's system showed never-ending, self-sustained chaos. While studying it, Lorenz uncovered two kinds of order within the chaos, now called strange attractors and iterated maps. Lorenz died, shortly after the lectures for this course had been recorded, at the age of 90.

Mandelbrot, Benoit (b. 1924). French-American mathematician, born in Poland. A polymath, he has analyzed the firing of nerve cells, the fluctuations of cotton prices, the shapes of galaxies and coastlines, the turbulence of fluids, and the dynamics of iterated maps. Much of his work has centered on the themes of irregularity, self-similarity, and heavy-tailed distributions, all of which are conspicuous in the natural world, yet all of which had been shunted off to the fringes of science as isolated curiosities. Synthesizing and extending work from many fields, he pulled these curiosities into a coherent framework that he called fractal geometry. Fractals also arise in the analysis of chaotic systems. In a sense they are the footprints of chaos, the geometric stamp of chaos.

May, Robert (b. 1936). Theoretical biologist. Born in Australia and trained as a theoretical physicist, his interests shifted early on to the dynamics of animal populations. His work in mathematical ecology is among the most highly regarded in the field. Later he turned his attention to the dynamics of infectious diseases. An elegant, witty speaker and writer, he published a review article in 1976 that brought the incredible nonlinear dynamics of the logistic map to a much wider scientific audience and helped to trigger the chaos boom that followed. He has been chief science adviser to the British government and is now Baron May of Oxford, a crossbencher in the House of Lords.

Newton, Isaac (1642–1727). Perhaps the greatest scientific genius of all time. Made unparalleled contributions to physics and mathematics. Invented calculus and differential equations. Postulated the law of universal gravitation and the three fundamental laws of motion. From them, he deduced that planets move in elliptical orbits and that they obey two other laws of planetary motion that Kepler had found empirically. As a person, he was difficult, secretive, and sexless. But the astonishing synthesis he created changed the world forever and ushered in the Scientific and Industrial Revolutions and the Enlightenment. Philosophically, his work led later thinkers to a supreme sense of confidence in the power of human reason.

Pecora, Louis (b. 1947). American physicist. Light-hearted and self-effacing, he was one of the first to see chaos as potentially useful, not just as an interesting curiosity. Working with his postdoctoral fellow Tom Carroll at the Naval Research Laboratory in Washington, DC, he used computer simulations and electrical circuits to demonstrate what others had thought impossible: Two identical chaotic systems could become perfectly synchronized while nevertheless fluctuating erratically. This work opened the door to using chaos for novel communication schemes.

Poincaré, Jules Henri (1854–1912). French mathematician. From a prominent family (cousin of Raymond, later president of the French Republic during World War I). The supreme mathematical mind of his era, comparable in sheer brainpower to Einstein, though less iconoclastic in spirit. He worked across the entire range of mathematics and physics, from geometry and topology to fluid mechanics and relativity theory. Intuitive and intensely geometric, yet unable to draw a decent sketch, he invented a new approach to differential equations, emphasizing the global behavior of all their solutions rather than focusing on special cases as others had done before him. While applying this approach to the greatest unsolved problem in astronomy—the three-body problem—he was shocked to discover the phenomenon we now call chaos.

Pollock, Jackson (1912–1956). American artist. Often classified as an Abstract Expressionist, he was inspired by the chaotic forces of nature and channeled them in his own work. In the 1940s he created a new technique, called drip painting. Pollock laid the canvas flat across the floor of his windswept barn and then walked around it, leaning in a state of controlled off-balance, while dripping, flinging, or pouring paint continuously onto the canvas. Though he was reviled as a drunk and a fraud by some critics at the time, his paintings are now valued at tens of millions of dollars.

Swinney, Harry (b. 1939). American experimental physicist. Interested in patterns, instabilities, and chaos in a wide range of nonlinear systems, from the Great Red Spot of Jupiter, to oscillating chemical reactions, to the flow of sand, ball bearings, and other granular materials. His 1975 experiments (with Jerry Gollub) on the onset of turbulence in fluid flow between two rotating cylinders lent early support to Ruelle and Takens's theory of turbulence based on strange attractors and decisively contradicted an older theory.

Taylor, Richard (b. 1963). Australian physicist, also trained in art theory. Works on applications of fractals to art and the visual sciences, as well as to electronic and optical nano-devices. He and his colleagues have applied fractal analysis to Jackson Pollock's drip paintings. They controversially claim that his paintings qualify as fractals and that their fractal dimension systematically increased over the decade while Pollock was refining his technique.

West, Geoffrey (b. 1940). Theoretical physicist. Born in a rural town in western England, he worked at Los Alamos as a particle physicist and then switched in midcareer to working on complex systems in biology and society. Currently president of the Santa Fe Institute, which is devoted to complexity studies. In 1997, he and his biological colleagues Jim Brown and Brian Enquist proposed a far-reaching explanation for Kleiber's law of metabolism and other scaling laws in biology, based on the pervasiveness of fractal branching networks in all living things.

Winfree, Arthur (1942–2002). American theoretical biologist. He worked on the dynamics of biological rhythms, from sleep-wake cycles to arrhythmias of the heart. Creative and exceptionally visual, his ideas and discoveries repeatedly opened new branches of study in mathematics, physics, and biology, for which he was awarded a MacArthur genius award (as his son Erik later was), along with top prizes in cardiology and applied mathematics. His first discovery, made as an undergraduate, concerned the collective synchronization of biological oscillators. He showed that as the coupling between the oscillators is increased, synchronization erupts spontaneously beyond a certain threshold, in a manner reminiscent of a phase transition.

Bibliography

Essential Reading:

Belbruno, E. *Fly Me to the Moon: An Insider's Guide to the New Science of Space Travel*. Princeton, NJ: Princeton University Press, 2007. A fun, fast-paced memoir by the creator of a new approach to space travel: surfing the gravitational chaos of the solar system.

Gleick, J. *Chaos: Making a New Science*. New York: Viking, 1987. The best book on chaos. It has everything—wonderful storytelling, memorable characters, an exhilarating sense of intellectual adventure, and exceptionally good explanations of the main ideas and why they matter. Read this book!

Lorenz, E. N. *The Essence of Chaos*. Seattle: University of Washington Press, 1993. Part memoir and part tutorial on the basics of chaos, this popular book by one of the giants of the field is characteristically understated, occasionally wry, and always illuminating.

Mandelbrot, B. *The Fractal Geometry of Nature*. San Francisco: W. H. Freeman, 1982. Idiosyncratic masterpiece by the genius who put fractals on the map. Hard to follow, but amazingly wide-ranging and original.

Peterson, I. *Newton's Clock: Chaos in the Solar System*. New York: W. H. Freeman, 1993. Excellent popular account of the development of celestial mechanics, written by a superb science journalist.

Rockmore, D. *Stalking the Riemann Hypothesis: The Quest to Find the Hidden Law of Prime Numbers*. New York: Pantheon, 2005. A terrific introduction to the Holy Grail of mathematics—the Riemann hypothesis—and its tantalizing connection to quantum chaos.

Schroeder, M. *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise*. New York: W. H. Freeman, 1991. Witty and erudite, this dazzling survey contains insights you won't find anywhere else. Aimed at readers comfortable with college-level mathematics.

Stewart, I. *Does God Play Dice? The Mathematics of Chaos*. Oxford: Blackwell, 1989. An outstanding popular account of chaos theory. Covers many of the same topics as Gleick's book, but in more mathematical depth and with less colorful stories.

Strogatz, S. *Sync: The Emerging Science of Spontaneous Order*. New York: Hyperion, 2003. Aimed at the general reader, this book explores nature's amazing ability to synchronize itself, from traffic patterns to brain waves.

Taylor, R. P. "Order in Pollock's Chaos." *Scientific American*, December 2002, 116–21. Taylor's excellent introduction to his fractal analysis of Jackson Pollock's drip paintings.

Whitfield, J. *In the Beat of a Heart: Life, Energy, and the Unity of Nature*. Washington, DC: Joseph Henry Press, 2006. Very nice book about the scaling laws of life and their proposed explanation by West, Brown, and Enquist in terms of the fractal networks inside all living things. Balanced, clear, and authoritative.

Recommended Reading:

Alon, U. *An Introduction to Systems Biology: Design Principles of Biological Circuits*. Boca Raton, FL: Chapman & Hall/CRC, 2006. An excellent text on the hot new field of systems biology, by one of its leaders.

Ball, P. *The Self-Made Tapestry: Pattern Formation in Nature*. New York: Oxford University Press, 1999. Offers broad coverage of patterns in nature, from fractal branching networks in plants and animals, to sand dunes, snowflakes, and zebra stripes.

Barabasi, A. L. *Linked: The New Science of Networks*. Cambridge, MA: Perseus, 2002. An engaging discussion of networks in science and society, by one of the pioneers in this emerging field.

Berry, M. "Quantum Physics on the Edge of Chaos." Chap. 15 in *Exploring Chaos: A Guide to the New Science of Disorder*, edited by N. Hall. New York: W. W. Norton, 1992. A fascinating introduction to the riddle of quantum chaos, by one of the world's most inventive theoretical physicists.

Buchanan, M. *Nexus: Small Worlds and the Groundbreaking Science of Networks*. New York: W. W. Norton, 2002. An excellent, wide-ranging introduction to the recent breakthroughs in the science of networks.

———. *Ubiquity: Why Catastrophes Happen*. New York: Three Rivers Press, 2001. A nice introduction to self-organized criticality, an ambitious and controversial theory that grew out of the chaos revolution of the 1970s and '80s. The goal was to explain why fractals and power laws are so ubiquitous and why so many diverse systems seem to organize themselves right to the brink of catastrophe (and sometimes beyond).

Cohen, I. B. *Science and the Founding Fathers: Science in the Political Thought of Jefferson, Franklin, Adams, and Madison*. New York: W. W. Norton, 1995. Especially intriguing for its discussion of Newton's impact on the thinking of Thomas Jefferson, as reflected in the language of the Declaration of Independence.

Crichton, M. *Jurassic Park*. New York: Knopf, 1990. Chaos theory is a central character in this bestselling novel about a theme park where cloned dinosaurs run amok, despite the best laid plans of the genetic engineers who brought them back to life.

Cvitanovic, P. *Universality in Chaos*. 2nd ed. Bristol, UK: Adam Hilger, 1989. A reprint collection of the seminal articles in chaos theory, up until the late 1980s. The famous theoretical papers of Lorenz, May, Feigenbaum, and Hénon are included, as are several experimental papers confirming the theory.

Diacu, F., and P. Holmes. *Celestial Encounters: The Origins of Chaos and Stability*. Princeton, NJ: Princeton University Press, 1996. The first chapter gives a great account of Poincaré's work on the three-body problem and his tortuous path to the discovery of chaos. Later chapters dive more deeply into the mathematics and discuss the ongoing interplay between chaos theory and celestial mechanics.

Ditto, W. L., and L. M. Pecora. "Mastering Chaos." *Scientific American*, August 1993, 78–84. Nicely explains how chaos can be harnessed for communications and used to stabilize erratically pulsating lasers and arrhythmic animal hearts.

Du Sautoy, M. *The Music of the Primes: Searching to Solve the Greatest Mystery in Mathematics*. New York: Harper Collins, 2003. Like Rockmore's book, this is an excellent popular account of the Riemann hypothesis of prime number theory and its intriguing connection to quantum chaos.

Feder, J. *Fractals*. New York: Plenum, 1988. A good technical monograph about the physics of fractals.

Frank, A. "Gravity's Rim." *Discover*, September 1994. Available online at http://discovermagazine.com/1994/sep/gravitysrim419. Entertaining popular treatment of Belbruno's work on using chaos in space travel.

Galison, P. *Einstein's Clocks, Poincaré's Maps: Empires of Time*. New York: W. W. Norton, 2003. The dramatic story of two scientific giants, Einstein and Poincaré, in a race to understand the nature of time. Gives a unique account of their scientific and technological milieu in the years just before the development of relativity theory. Galison nicely sets Poincaré's work on chaos in this larger context.

Garfinkel, A., M. L. Spano, W. L. Ditto, and J. N. Weiss. "Controlling Cardiac Chaos." *Science*, August 28, 1992, 1230–35. A demonstration that drug-induced arrhythmia in a rabbit heart can be stabilized by intermittent, gentle electrical stimulation applied at specific times dictated by chaos theory.

Girardot, N. J. *Myth and Meaning in Early Taoism: The Theme of Chaos (Hun-Tun)*. Los Angeles: University of California Press, 1983. Explores the Taoist concept of chaos, which foreshadowed many of our contemporary scientific notions.

Glass, L., and M. C. Mackey. *From Clocks to Chaos: The Rhythms of Life*. Princeton, NJ: Princeton University Press, 1988. Very accessible monograph on biological oscillations and chaos.

Goldberger, A. L., D. R. Rigney, and B. J. West. "Chaos and Fractals in Human Physiology." *Scientific American*, February 1990, 42–49. Popular account of the role that chaos and fractals play in human health and disease.

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Hall, N., ed. *Exploring Chaos: A Guide to the New Science of Disorder*. New York: W. W. Norton, 1992. A collection of excellent expository articles about chaos, reprinted from the British magazine *New Scientist* and written by such luminaries as Robert May, Benoit Mandelbrot, and Michael Berry.

Iasemidis, L. D., et al. "Adaptive Epileptic Seizure Prediction System." *IEEE Transactions on Biomedical Engineering* 50 (2003): 616–27. Technical article showing that there are recognizable precursors to epileptic seizures, as revealed by analysis of a patient's EEG (brain waves) in the minutes before the seizure. The article goes on to describe a seizure prediction algorithm that is genuinely predictive, meaning that it warns of impending seizures and is not merely based on retrospective data.

Klarreich, E. "Navigating Celestial Currents: Math Leads Spacecraft on Joy Rides Through the Solar System." *Science News*, April 16, 2005, 250–52. Excellent account of how chaos can be used to send unmanned probes through the solar system at very low cost, starting with Belbruno's early work and continuing to the more recent concept of an interplanetary superhighway, now used by NASA for some of its missions.

Kline, M. *Mathematics in Western Culture*. New York: Oxford University Press, 1953. A magnificent book, feisty and learned and full of insight. Kline persuasively documents the importance of math in all aspects of Western culture. The chapters about Newton's influence on science, philosophy, religion, literature, and aesthetics are wonderful, but so is the rest of the book.

Liebovitch, L. S. *Fractals and Chaos Simplified for the Life Sciences*. New York: Oxford University Press, 1998. A pedagogically creative text, organized in an unusual way: The pages come in pairs, with crisp explanations of basic concepts on the left-hand page accompanied by real data or line drawings on the right-hand page. Includes good discussions of fractal statistics, fractal processes in biology and their significance, the medical benefits of chaos, and techniques for controlling chaos.

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Mandelbrot, B., and R. L. Hudson. *The (Mis)behavior of Markets: A Fractal View of Risk, Ruin, and Reward*. New York: Basic Books, 2004. An iconoclastic, fractal approach to finance.

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Peak, D., and M. Frame. *Chaos Under Control: The Art and Science of Complexity*. New York: W. H. Freeman, 1994. Lively text about chaos and fractals. Unusual for its broad coverage, including connections to music, art, and fiction.

Peitgen, H. O., and P. H. Richter. *The Beauty of Fractals: Images of Complex Dynamical Systems*. Berlin: Springer-Verlag, 1986. The stunning multicolor images of the Mandelbrot set, now commonplace, first appeared here. Still great as a coffee-table book, though its mathematics is presented at an advanced level.

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Stoppard, T. *Arcadia*. New York: Faber and Faber, 1993. Improbable as it may sound, Tom Stoppard managed to weave beautifully clear explications of chaos, fractals, iterated maps, and entropy into this hit play about the mysteries of time, determinism, and love.

Strogatz, S. H. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*. Cambridge, MA: Perseus, 1994. The most widely used textbook on chaos theory. Emphasizes applications, concrete examples, and geometric intuition. Assumes knowledge of first-year calculus and physics.

Strogatz, S. H., D. M. Abrams, A. McRobie, B. Eckhardt, and E. Ott. "Crowd Synchrony on the Millennium Bridge." *Nature*, November 3, 2005, 43–44. A mathematical model of what happened to the Millennium Bridge on opening day.

Taleb, N. N. *The Black Swan: The Impact of the Highly Improbable*. New York: Random House, 2007. A lively (and sometimes comically belligerent) book about the way the world really works. Taleb argues that life is far more random than we are willing to admit and that history is dominated by rare, unforeseeable events that change everything.

Turcotte, D. L. *Fractals and Chaos in Geology and Geophysics*. 2nd ed. Cambridge: Cambridge University Press, 1997. Authoritative treatment of fractals and power laws applied to earthquakes, wildfires, landscape topography, and other phenomena in the Earth sciences, by a leader in this field.

Winfree, A. T. *The Timing of Biological Clocks*. New York: Scientific American Library, W. H. Freeman, 1987. A fascinating, sometimes quirky survey of daily rhythms in plants and animals, written by an exceptionally creative scientist. Winfree uses visual mathematics (specifically, mappings between circles) and multicolored pictures to reveal the underlying principles of biological oscillations, and in this way exposes a gorgeous unity among diverse living things.

———. *When Time Breaks Down: The Three-Dimensional Dynamics of Electrochemical Waves and Cardiac Arrhythmias*. Princeton, NJ: Princeton University Press, 1987. Winfree was also a pioneer in the use of mathematics to explain why biological rhythms sometimes stop abruptly. Here he explores the mystery of sudden cardiac death, which kills hundreds of thousands of seemingly healthy people each year, when their hearts lapse into fibrillation, a mysterious kind of electrical turbulence.

Internet Resources:

Eric Heller's online gallery of quantum art: www.ericjhellergallery.com.
Frequently asked questions about chaos (somewhat out of date, but still valuable): http://www.faqs.org/faqs/sci/nonlinear-faq/.

A fascinating conversation with the playwright Tom Stoppard about the starring role of chaos theory in his hit play *Arcadia*: http://www.siam.org/news/news.php?id=727.

For all things fractal: http://classes.yale.edu/Fractals/.

Online simulations (Java applets) for chaos and fractals: http://math.bu.edu/DYSYS/applets/.

Excellent compendium of Java applets for chaos and fractals: http://www.student.math.uwaterloo.ca/~pmat370/JavaLinks.html.

Advanced software for dynamical systems (mostly for mathematically sophisticated users): http://www.dynamicalsystems.org/sw/sw/.

Advanced tutorials about dynamical systems, many accompanied by online simulations: http://www.dynamicalsystems.org/tu/tu/.

Laboratory demonstrations of chaos and other nonlinear phenomena—a video hosted by Professor Strogatz, with the help of several colleagues: http://ecommons.library.cornell.edu/handle/1813/97.

Java applets for billiards: http://serendip.brynmawr.edu/chaos/home.html.

More links for chaos, fractals, nonlinear dynamics, etc.: http://mathforum.org/library/topics/dynamical_systems/.

For applets about many fascinating complex systems: http://ccl.northwestern.edu/netlogo/models/.

Website about the Millennium Bridge: http://www.arup.com/MillenniumBridge/.

Explore the statistical distribution of earthquakes of various magnitudes: http://www.data.scec.org/Module/s2act08.html.

Internet Resources by Lecture:

Lecture Four

You can play with a simulation of a pendulum and its motion in state space here: http://www.aw-bc.com/ide/idefiles/media/JavaTools/pndulums.html.

Lecture Five

If you want to examine the butterfly effect for yourself in a simulation of another of Lorenz's models (to be discussed in Lectures Seven and Eight), try this online Java applet: http://www.aw-bc.com/ide/idefiles/media/JavaTools/lrnzdscv.html.

Lecture Seven

Play with a Java applet of Lorenz's attractor here: http://www.aw-bc.com/ide/idefiles/media/JavaTools/lrnzr320.html.

For a simulation that ties the mathematical model to the physical problem of convection, try this one: http://www.aw-bc.com/ide/idefiles/media/JavaTools/lrnzphsp.html.

Lecture Eight

Play with Lorenz's iterated map at this site: http://www.aw-bc.com/ide/idefiles/media/JavaTools/lrnzzmax.html.

Lecture Nine

Explore the dynamics of the logistic map yourself. Go to: http://www.student.math.uwaterloo.ca/~pmat370/JavaLinks.html to find some relevant Java applets.

One of the simplest to use is this one: http://www.geom.uiuc.edu/~math5337/ds/applets/iteration/Iteration.html.

Lecture Ten

Explore the orbit diagram yourself. A very nice Java applet is here: http://math.bu.edu/DYSYS/applets/bif-dgm/Logistic.html. Try to find the mini-orbit diagrams at the end of a big periodic window.

Lecture Thirteen

Play with Hénon's mapping and strange attractor here: http://www.cmp.caltech.edu/~mcc/Chaos_Course/Lesson5/Demo1.html.

Lecture Twenty-Two

Explore the dynamics of billiards on tables of various shapes here: http://serendip.brynmawr.edu/chaos/home.html.

Eric Heller's online gallery of quantum art: www.ericjhellergallery.com.

Lecture Twenty-Three

Play with a model for the synchronization of fireflies, using this applet: http://ccl.northwestern.edu/netlogo/models/Fireflies.